Nov 4, 2009.

Areas and lengths in Polar Coordinates.

Given a little arc $r = f(\theta)$ between θ and $\theta + \delta\theta$, the area of the wedge from the origin to the arc is roughly $\frac{\pi r^2}{2\pi}\delta\theta$ so that of a wedge formed by $r = f(\theta)$ between θ_1 and θ_2 is

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} [f(\theta)]^2 d\theta$$

The arc length on the other hand is computed by

$$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$$

and

$$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta, \quad y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta, y'(\theta)$$

$$[x'(\theta)]^2 + [y'(\theta)]^2 = [f'(\theta)]^2 + [f(\theta)]^2$$

so that the length of the segment $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is

$$\int_a^b \left[[f'(\theta)]^2 + [f(\theta)]^2\right]^{\frac{1}{2}} d\theta$$

Example: The semi-circle $(x-1)^2 + y^2 = 1; y \geq 0$ is in polar coordinates $r = 2\sin\theta,\ 0 \leq \theta, \frac{\pi}{2}$. Its area is

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} 4\sin^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta)\, d\theta = \frac{\pi}{2}$$

Its length is

$$\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2\theta + 4\cos^2\theta}\, d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \pi$$

The full circle is obtained by θ ranging over $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This will give a length of 2π, and an area of π.

Homework.

Q1. Find the length and the Area of the Cardioid $r = 1 + \cos\theta$.

Q2. Sketch the curve $r = \sin 6\,\theta; 0 \leq \theta \leq \pi$. How many loops does it have? What is the area of each loop?

Monday's quiz will be on Areas and arc lengths in polar coordinates.