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Taylor Expansion.

A function $f(x)$ that has n derivatives can be written as

$$\begin{aligned} f(x) &= f(a) + f^{(1)}(a)(x-a) + \frac{1}{2}f^{(2)}(a)(x-a)^2 + \cdots + f^{(n)}(a)(x-a)^n + R_n(x) \\ &= P_n(x-a) + R_n(x) \end{aligned}$$

How big is $R_n(x)$? $\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0$. See this by L'Hospital's rule.

$$R_n(a) = R_n^{(1)}(a) = \cdots = R_n^{(n)}(a) = 0$$

$$\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = \frac{R_n^{(n)}(a)}{n!} = 0$$

Convergent Taylor series is when $R_n(x) \to 0$ if $|x-a| < R$.

Examples; 1. $f(x) = \cos x$. $f'(x) = -\sin x$, $f''(x) = -\cos x$. $a = \frac{\pi}{4}$. $\sin a = \cos a = \frac{1}{\sqrt{2}}$.

$$\cos(\frac{\pi}{4} + \frac{1}{10}) = \cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4})\frac{1}{10} - \frac{1}{2}\cos(\frac{\pi}{4})(\frac{1}{10})^2 + Error. \simeq \frac{1}{\sqrt{2}}[1 - 0.1 - 0.005]$$

MacLaurin Expansion is when $a = 0$. For instance

$$\cos\frac{1}{10} = \cos 0 - \sin 0\frac{1}{10} - \frac{1}{2}\cos 0(\frac{1}{10})^2 = 1 - .005 \simeq 0.995$$

Homework

Evaluate the following by Taylor-Maclaurin expansion using two derivatives.

1. $(27.1)^{\frac{1}{3}}$ $x^{\frac{1}{3}}$ around $x = 27$ i.e. $a = 27$

2. $\sin 31($ in degrees$)$. Expand around $\frac{\pi}{6}$.

For what values of x will the following power series converge?

3.

$$\sum_{n=1}^{\infty} n!x^n$$

4.

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3}$$

5.

$$\sum_{n=0}^{\infty} 2^n x^n$$