Oct 14, 2009.

Power Series.

$$f(x) = \sum_n a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

It may not converge. It always converges when $x = 0$. If it converges for some value say $x = x_0$, then $a_n x_0^n \to 0$. Therefore $|a_n x_0^n| \leq C$. If $|x| < |x_0|$ by comparison

$$|a_n x^n| \leq |\frac{x}{x_0}|^n |a_n x_0| \leq C|\frac{x}{x_0}|^n$$

which converges. Either the series converges for all x, or only for $x = 0$ or for $|x| < R$ and not for $|x| > R$. If $x = \pm R$ it is not clear.

Examples.

1. $\sum \frac{x^n}{n!}$. Converges for all x. Do ratio test. $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0$

2. $\sum x^n$ converges if $|x| < 1$ but not if $x = \pm 1$ or $|x| > 1$.

3. $\sum \frac{x^n}{n}$ converges if $|x| < 1$, diverges if $|x| > 1$. $x = -1$ is OK but not $x = 1$.

4. $\sum \frac{x^n}{n^2}$ converges if $|x| \leq 1$ and diverges if $|x| > 1$.

5. $\sum_n n! x^n$ converges only if $x = 0$.

If

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

then $f(0) = a_0, f'(0) = a_1, f''(0) = 2a_2, \ldots, f^{(n)}(0) = n! a_n$. One can start with a function f with all derivatives and write the series

$$f(x) = \sum_n f^{(n)}(0) \frac{x^n}{n!}$$

The series may converge for $|x| < R$ and diverge for $|x| > R$. Even when the series converges there is guarantee that that the sum is $f(x)$. More terms give better approximations near 0 but not away from 0. If $|f^{(n)}(x)| \leq C_n$ on $0 \leq x \leq R$, then

$$|f(x) - \sum_{j=1}^{n-1} f^{(j)}(0) \frac{x^j}{j!}| \leq C_n \frac{R^n}{n!}$$

We have to estimate the error and show it goes to zero.

Examples.

1. $f(x)=e^x$. $f^{(n)}(x)=e^x$. $f^{(n)}(0)=1$. $|f^{(n)}(x)|=e^R$. $\frac{e^R R^n}{n!}\to 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

2. $f(x)=\frac{1}{1-x}$. Geometric series

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$$

Integrate and differentiate term by term.

$$\frac{1}{(1-x)^2} = \sum_{j=0}^{\infty} j x^{j-1} = 1 + 2x + 3x^2 + \cdots$$

Change x to $-x$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

Integrate. Constant term is equal to $\log(1+0)=\log 1 = 0$.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

In particular

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

Repalce x by x^2

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

Integrate. Constant term is again $\arctan 0 = 0$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}$$

In particular

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1}$$