Things to remember from Calculus 1.

Derivatives:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>0</td>
</tr>
<tr>
<td>$x^p$</td>
<td>$px^{p-1}$</td>
</tr>
<tr>
<td>$\log x$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$\cos x$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$-\sin x$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$\sec^2 x$</td>
</tr>
<tr>
<td>$\sec x$</td>
<td>$\sec x \tan x$</td>
</tr>
<tr>
<td>$\csc x$</td>
<td>$-\csc x \cot x$</td>
</tr>
<tr>
<td>$\cot x$</td>
<td>$-\csc^2 x$</td>
</tr>
<tr>
<td>$\arcsin x$</td>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\arccos x$</td>
<td>$-\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\arctan x$</td>
<td>$\frac{1}{1+x^2}$</td>
</tr>
</tbody>
</table>

Product Rule:

$f(x)g(x)$  \hspace{1cm} $f'(x)g(x) + f(x)g'(x)$

Quotient Rule:

$\frac{f(x)}{g(x)}$  \hspace{1cm} $\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule:

$f(g(x))$  \hspace{1cm} $f'(g(x))g'(x)$

Exercise: Derive all the trigonometric relations from just one. The derivative of $\sin x$ is $\cos x$.

$$(\cos x)' = [\sin(\frac{\pi}{2} - x)]' = [\cos(\frac{\pi}{2} - x)] \times (\frac{\pi}{2} - x)' = [\sin x] \times (-1) = -\sin x$$

$$(\tan x)' = \left[\frac{\sin x}{\cos x}\right]' = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x}$$

$$= \frac{(\cos x)^2 + (\sin x)^2}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$
Anti-derivatives, Indefinite Integrals.

\[
\begin{align*}
f'(x) & \quad f(x) + C \\
\int f'(x) \, dx & = f(x) + C \\
\int x^k \, dx & = \frac{x^{k+1}}{k+1} + C \quad \text{if } k \neq 1 \\
\int \frac{1}{x} \, dx & = \log x + C \\
\int \cos x \, dx & = \sin x + C \\
\int \sin x \, dx & = -\cos x + C \\
\int \tan x \, dx & = -\log \cos x + C \\
\int \frac{1}{\sqrt{1-x^2}} \, dx & = \arcsin x + C \\
\int \frac{1}{1+x^2} \, dx & = \arctan x + C \\
\int e^x \, dx & = e^x + C \\
\int \frac{f'(x)}{f(x)} \, dx & = \log f(x) + C \\
\int f(x)^k f'(x) \, dx & = \frac{f(x)^{k+1}}{k+1} + C \\
\int f'(g(x))g'(x) \, dx & = f(g(x)) + C
\end{align*}
\]

Substitution.

\[
\int f(x) \, dx = F(x) + C; \quad x = g(y); \quad \int f(g(y)) \frac{dx}{dy} \, dy = \int f(g(y))g'(y) \, dy = F(g(y)) + C
\]

Examples:

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx
\]

Put \( x = \sin y \).

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \int \frac{\cos y}{\sqrt{1-\sin^2 y}} \, dy = \int 1 \, dy = y + C = \arcsin x + C
\]

\[
\int \frac{1}{1+x^2} \, dx
\]

\[
2
\]
Put $x = \tan y$.

\[
\int \frac{1}{1 + x^2} dx = \int \frac{\sec^2 y}{1 + \tan^2 y} dy = \int 1 dy = y + C = \arctan x + C
\]

In general for $\int f(\sqrt{1 - x^2}) dx$ try $x = \sin y$ and for $\int f(\sqrt{1 + x^2})$ try $x = \tan y$.

Integration by parts.

\[
\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx
\]

or

\[
\int u(x) dv(x) = u(x)v(x) - \int v(x) du(x)
\]

Examples:

\[
\int \log x dx = \int \log x \cdot 1 dx = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - x + C
\]

\[
\int x \sin x dx = - \int x \cos x = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C
\]

You can integrate by parts repeatedly.

\[
\int x^2 e^x dx = \int x^2 e^x dx = x^2 e^x - 2 \int xe^x dx = x^2 e^x - 2xe^x + 2 \int e^x dx = (x^2 - 2x + 2)e^x + C
\]

Definite Integrals. Fundamental Theorem of Calculus. Area under a curve.

If $y = f(x)$ is a curve (assume $f(x) > 0$) then the area $A$ between the curve and the $x$-axis between the ordinates $x = a$ and $x = b$ can be obtained as

\[
A = \int_{a}^{b} f(x)dx = F(b) - F(a)
\]

where $F'(x) = f(x)$. In particular if

\[
A(y) = \int_{a}^{y} f(x)dx
\]

then

\[
A'(y) = f(y)
\]

If $f$ is not positive then the area above $x$-axis is positive and the area below is negative. Then the sum is what can still be computed as $\int_{a}^{b} f(x)dx$. For example,

\[
\int_{-1}^{0} x dx = \frac{x^2}{2} \bigg|_{-1}^{0} = -\frac{1}{2}
\]
\[
\int_0^1 x \, dx = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2}
\]

and
\[
\int_{-1}^1 x \, dx = \left. \frac{x^2}{2} \right|_{-1}^1 = 0
\]

By convention to be consistent we define for \( a > b \),
\[
\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx
\]

so that for any \( a, b \)
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

Examples: Area inside a circle of radius \( r \).

\[
A(r) = 4 \int_0^r \sqrt{r^2 - x^2} \, dx
\]

Put \( x = r \sin y \). \( dx = r \cos y \, dy \). Change limits. when \( x = 0, y = 0 \). When \( x = r, y = \frac{\pi}{2} \).

\[
A(r) = 4 \int_0^{\frac{\pi}{2}} r \cos y \cdot r \cos y \, dy = 4 r^2 \int_0^{\frac{\pi}{2}} \cos^2 y \, dy = 2 r^2 \int_0^{\frac{\pi}{2}} (1 - \cos 2y) \, dy
\]

\[
= 2r^2 \left[ y - \sin 2y \frac{y}{2} \right]_0^{\frac{\pi}{2}} = \pi r^2
\]

Why should you change limits? Chain rule says

\[
[f(g(x))]' = f'(g(x))g'(x)
\]

In particular
\[
\int_a^b [f(g(x))]' \, dx = f(g(b)) - f(g(a))
\]

If we change variables by \( y = g(x) \) the integral becomes
\[
\int f'(y) \, dy = f(y) + C
\]

And
\[
\int_{g(a)}^{g(b)} f'(y) \, dy = f(g(b)) - f(g(a))
\]

is the right answer.
Partial Fractions. This involves rewriting the function $f$ to be integrated in a different form to make it easier.

Examples:

\[ \int \frac{1}{(x+1)(x+2)}\,dx \]

Write

\[ \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} \]

Then

\[ \int \frac{1}{(x+1)(x+2)}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx = \log(x+1) - \log(x+2) + C \]

Another Example.

\[ \int \frac{3x + 4}{(x+1)(x+2)}\,dx \]

\[ \frac{3x + 4}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2} = \frac{(a + b)x + (2a + b)}{(x+1)(x+2)} \]

\[ a + b = 3, \quad 2a + b = 4 \]
\[ a = 1, \quad b = 2 \]

\[ \frac{3x + 4}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{2}{x+2} \]

\[ \int \frac{3x + 4}{(x+1)(x+2)}\,dx = \log(x+1) + 2\log(x+2) + C \]