Vortex stretching in incompressible and compressible fluids

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1 Introduction

The primitive form of the incompressible Euler equations is given by

$$\frac{du}{dt} = u_t + u \cdot \nabla u = -\nabla \left(\frac{P}{\rho} + gz \right) \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

representing conservation of momentum and mass respectively. Here u is the velocity vector, P the pressure, ρ the constant density, g the acceleration of gravity and z the vertical coordinate. In this form, the physical meaning of the equations is very clear and intuitive. An alternative formulation may be written in terms of the vorticity vector

$$\omega = \nabla \times u \,, \tag{3}$$

namely

$$\frac{d\omega}{dt} = \omega_t + u \cdot \nabla \omega = (\omega \cdot \nabla) u \tag{4}$$

where u is determined from ω nonlocally, through the solution of the elliptic system given by (2) and (3). A similar formulation applies to smooth isentropic compressible flows, if one replaces the vorticity ω in (4) by ω/ρ.

This formulation is very convenient for many theoretical purposes, as well as for better understanding a variety of fluid phenomena. At an intuitive level, it reflects the fact that *rotation* is a fundamental element of fluid flow, as exemplified by hurricanes, tornados and the swirling of water near a bathtub sink. Its derivation from the primitive form of the equations, however, often relies on complex vector identities, which render obscure the intuitive meaning of (4). My purpose here is to present a more intuitive derivation, which follows the traditional physical wisdom of looking for integral principles first, and only then deriving their corresponding differential form. The integral principles associated to (4) are conservation of mass, circulation–angular momentum (Kelvin's theorem) and vortex filaments (Helmholtz' theorem). These principles imply a "stretching" mechanism for the magnitude of ω and an advective mechanism for its direction which have (4) as their differential expression.

Finally, we shall derive a scalar version of the vector identity (4), which remains valid even for non isentropic compressible flows and incompressible flows with nonconstant density. This is the conservation of the potential vorticity

$$q = \frac{\omega \cdot \nabla S}{\rho}\,, \tag{5}$$

first introduced by Ertel. Here S denotes any quantity convected by the flow; the entropy is a good choice for compressible fluids, as is the density for incompressible ones. Ertel's theorem states that

$$\frac{dq}{dt} = 0\,, \tag{6}$$

so the potential vorticity is convected by the flow.

2 Vortex Tubes and Circulation

Since ω is the curl of a vector field, its divergence is zero. Therefore, it is natural to consider vortex tubes, analogous to the stream tubes of a divergence free velocity field. We start with vortex lines which, analogous to stream lines, are defined, at each fixed time, as integral lines of the vorticity field (that is, lines which are everywhere tangent to the vorticity vector.) Next we consider a closed contour in R^3, and define as a vortex tube the set of all vortex lines which intersect the contour.

Consider two cross-sections of a vortex tube. Since the vorticity is everywhere tangent to the surface of the tube, and has zero divergence, the divergence theorem tells us that the normal flux of vorticity through the two cross-sections must be equal. This normal flux is therefore a constant; it is called the strength of the vortex tube.

Stokes' theorem equates the normal flux of vorticity through a cross-section of a vortex tube with the line integral of tangential velocity around it. Therefore this quantity, called circulation and denoted Γ, is a constant along the tube, equal to its strength. In symbols,

$$\Gamma = \int_C u \cdot dr = \int_S \omega_n \, dS = \text{ constant along a vortex tube,}$$

where S is an arbitrary cross-section of the tube and C its perimeter. A fundamental property of inviscid flows is that Γ does not change in time following a contour C which moves with the fluid. This is the content of Kelvin's theorem, which we prove in the following section.

3 Kelvin's Theorem

Kelvin's theorem is equivalent to the statement of conservation of angular momentum for a closed fluid filament. Since, for inviscid flows, all forces acting on a filament are normal to it, no net torque is applied, and angular momentum is preserved. The following is an alternative, more formal derivation:

Consider a closed contour $C(t)$ which moves with the fluid. The position of each of its constitutive points may be written as $r(s,t)$, where s is any parameter along the contour, and t is the time. The property of C moving with the fluid takes the form

$$\frac{\partial r(s,t)}{\partial t} = u(r(s,t),t)\,. \tag{7}$$

The time derivative of the circulation Γ along $C(t)$ is given by

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int u(r(s,t),t)\cdot\frac{\partial r(s,t)}{\partial s}\,ds = \int \frac{du}{dt}\cdot r_s\,ds + \int u\cdot r_{st}\,ds \tag{8}$$

In the first integral of the right hand side of (8),

$$\frac{du}{dt} = u_t + r_t\cdot\nabla u = u_t + u\cdot\nabla u = -\nabla\left(\frac{P}{\rho} + gz\right)$$

Therefore

$$\int \frac{du}{dt}\cdot r_s\,ds = -\int \nabla\left(\frac{P}{\rho}+gz\right)\cdot dr = -\int d\left(\frac{P}{\rho}+gz\right) = 0\,,$$

since we are integrating along a closed contour. As for the second integral in the right hand side of (8), we have

$$\int u\cdot r_{st}\,ds = \int u\cdot u_s\,ds = \int u\cdot du = \int d\left(\frac{u^2}{2}\right) = 0\,.$$

Therefore

$$\frac{d\Gamma}{dt} = 0 \tag{9}$$

which is the statement of Kelvin's theorem. Before we can fully interpret this theorem in physical terms, we need to show an important corollary, known as Helmholtz' theorem: Vortex tubes move with the fluid.

4 Helmholtz' Theorem

Consider a surface which, at time $t = t_0$, is a vortex tube. As time progresses, the particles which constitute this surface will move, so the surface will deform. Nevertheless, the surface remains a vortex tube throughout its deformation. This is a simple corollary of Kelvin's theorem. In order to prove it, consider the surface at a later time, and pick any closed contour in it homotopic to a point (that is, a contour which does not turn around the tube.) By Kelvin's theorem, the circulation around this contour has the same value it had at time $t = t_0$, i.e. zero, since at that time the surface was tangent to the vorticity field. Since this argument applies to an arbitrary contour on the surface, we conclude that the vorticity field is still tangent to it; so the surface is still a vortex tube.

Helmholtz' name is customarily associated with the application of this theorem to vortex filaments, vortex tubes with negligible cross-sectional area which we may identify with vortex lines: Vortex filaments move with the fluid. This is a fundamental constraint on the evolution of the direction of vorticity.

5 The Vortex-Stretching Mechanism

Conservation of mass and Kelvin and Helmholtz' theorems combined provide the grounds for one of the most important mechanisms of fluid flow: the stretching of vortex tubes. Consider a vortex tube immersed in a fluid. The circulation around it (the tube's strength) is a constant along the tube's length, which implies that the absolute value of vorticity is largest where the cross-sectional area of the tube is smallest. This is analogous to the fluid's acceleration at a contraction in a stream tube, which follows from conservation of mass.

Consider now the time evolution of a vortex tube. According to Helmholtz' theorem, tubes move with the fluid. Moreover, from Kelvin's theorem, their strength does not change with time. Therefore, if the area of a cross-section of a tube should become very small, vorticity would have to amplify proportionally. Since the fluid is incompressible, however, the volume between two sections of the tube remains constant. Therefore any shrinking of the cross-sectional area must be accompanied by a longitudinal stretching. We conclude that the local stretching of a vortex tube gives rise to a proportional amplification of the absolute value of vorticity. This is the so called vortex-stretching mechanism.

The principle of the previous paragraph applies to the mean value of the vorticity normal to a cross-section of vortex tubes. When applied to a vortex filament, however, it becomes a precise statement about the local magnitude of the vorticity vector: When a vortex filament stretches, the intensity of vorticity grows proportionally. Since, moreover, the vorticity vector remains always tangent to the vortex filament, we may say that the vorticity vector is proportional to the element of filament to which it is attached, with a proportionality constant which does not depend on time. In symbols, if we describe a vortex filament by specifying the position of each of its constitutive particles $r(s,t)$, where s is a parameter along the filament, the following relation holds:

$$w(r(s,t),t) = c(s)\frac{\partial}{\partial s}r(s,t) \tag{10}$$

where $c(s)$ is a scalar function of s which does not depend on time. Equation (4) follows immediately from (10):

$$\frac{d\omega}{dt} = c(s)\frac{\partial}{\partial s}u(r(s,t),t) = c(s)\frac{\partial}{\partial s}r(s,t)\cdot\nabla u(r(s,t),t) = (\omega\cdot\nabla)u$$

This equation therefore is the differential expression of Kelvin and Helmholtz' theorems for an incompressible fluid.

6 Further Formalization

Here we shall provide a more formal derivation of equation (10) than the one described above. For starters, we shall do a "semi-formal" derivation, using "very thin" vortex tubes, which we will formalize further afterwards. Helmholtz' theorem tells us that vortex filaments move with the fluid; therefore, for a vortex filament described by its position $r(s,t)$, the following identity holds:

$$w(r(s,t),t) = c(s,t)\frac{\partial}{\partial s}r(s,t)\,, \tag{11}$$

where $c(s,t)$ is a scalar. Equation (11) is merely a rewriting of the definition of a vortex filament, which is a curve everywhere tangent to the vorticity field. If we manage to show that c does not depend on t, we will obtain (10).

To this end, consider a very thin vortex tube, which we will "parametrize" using one of its filaments $r(s,t)$ and the cross-sectional area $\Omega(s,t)$. The strength Γ of the tube, which is independent of s and t, due to Kelvin's theorem, is given approximately by

$$\Gamma = \omega(r(s,t),t)\cdot\Omega(s,t) = c(s,t)\frac{\partial}{\partial s}r(s,t)\cdot\Omega(s,t) \tag{12}$$

On the other hand, the volume of fluid between two sections of the tube, also a constant, due to conservation of mass, is

$$V = \int_{s_1}^{s_2}\frac{\partial}{\partial s}r(s,t)\cdot\Omega(s,t)\,ds \tag{13}$$

which, using (12), may be written as

$$V = \Gamma\int_{s_1}^{s_2}\frac{ds}{c(s,t)}$$

Since V cannot depend on time, we conclude that c does not either, which ends the proof.

To formalize this proof further, we need to get rid of the concept of "very thin" tubes, and the corresponding approximate identities (12) and (13). To this end, consider a vortex tube with finite size. The filaments within the tube may be described by the function

$$r(s,\alpha,t)$$

where α is a two-dimensional parameter which selects a filament. Then, from Helmholtz' theorem,

$$w(r(s,\alpha,t),t) = c(s,\alpha,t)\frac{\partial}{\partial s}r(s,\alpha,t)$$

Next we shall mimic the arguments above, for cross-sections of the tube defined by the condition $s =$ constant. The flux through one such section may be written as

$$\Gamma = \int_\alpha \omega(r(s,\alpha,t),t) \cdot d\Omega(s,\alpha,t) = \int_\alpha c(s,\alpha,t) \frac{\partial}{\partial s} r(s,\alpha,t) \cdot d\Omega(s,\alpha,t)\,. \quad (14)$$

The vector $d\Omega$ is the oriented differential area of the cross-section, which may be expressed formally as

$$d\Omega = \Omega_\alpha(s,\alpha,t)\, d\alpha_1 d\alpha_2\,,$$

where

$$\Omega_\alpha(s,\alpha,t) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial r_1}{\partial \alpha_1} & \frac{\partial r_2}{\partial \alpha_1} & \frac{\partial r_3}{\partial \alpha_1} \\ \frac{\partial r_1}{\partial \alpha_2} & \frac{\partial r_2}{\partial \alpha_2} & \frac{\partial r_3}{\partial \alpha_2} \end{bmatrix}.$$

(The subindices refer to the components of the vectors r and α.) Since Γ must be a constant for arbitrary widths of the tube, we conclude that the quantity

$$\Gamma_\alpha(\alpha) = c(s,\alpha,t) \frac{\partial}{\partial s} r(s,\alpha,t) \cdot \Omega_\alpha(s,\alpha,t)$$

is independent of s and t. The volume between two sections, on the other hand, is given by

$$V = \int_{s_1}^{s_2} \int_\alpha \frac{\partial}{\partial s} r(s,\alpha,t) \cdot d\Omega(s,\alpha,t)\, ds = \int_{s_1}^{s_2} \int_\alpha \frac{\Gamma_\alpha(\alpha)\, d\alpha_1 d\alpha_2}{c(s,\alpha,t)}\, ds\,. \quad (15)$$

Since this volume must be time independent, we conclude that c does not depend on t.

7 Kelvin and Helmholtz Theorems for Compressible Flows

Many of the ideas developed above extend with small variations to compressible flows. Here we shall see which these variations are, and under which conditions they apply. The first important constraint is that the flow should be smooth, i.e. shock-free, since vorticity can be created at shocks. Therefore we will concentrate here on smooth solutions, and we will write the Euler equations for compressible fluids in the following non-conservative form:

$$\begin{aligned} \rho_t + \nabla \cdot (\rho u) &= 0 && (16) \\ u_t + u \cdot \nabla u &= -\frac{\nabla P}{\rho} + \nabla(gz) && (17) \\ S_t + u \cdot \nabla S &= 0\,, && (18) \end{aligned}$$

where ρ is the fluid density, u the vector velocity and S the entropy, defined by

$$TdS = de + Pd\left(\frac{1}{\rho}\right). \tag{19}$$

Here T is the absolute temperature and e the internal energy of the gas. The right hand side of (19) measures the energy absorption by an element of fluid, divided into internal energy and work performed on its surroundings. The inverse of the temperature provides an integrating factor for this differential; the corresponding integral is the entropy S.

If we recall the derivation of Kelvin's theorem for incompressible flows, the fact that the right-hand side of the momentum equation (1) could be written as a gradient played a crucial role. This will only be the case for equation (17) if either the density is a constant or the pressure P is a function of ρ alone. Now, in general, the pressure is a function of the density and the entropy, i.e.

$$P = P(\rho, S).$$

However, it is clear from equation (18) that, if the entropy is initially a constant (as it is for a uniform fluid at rest), it will remain constant forever –provided, of course, that no shocks are formed. Thus we will restrict our attention to this case; i.e. we shall consider only *isentropic flows*. For these, the right-hand side of (17) can be written as

$$-\frac{\nabla P}{\rho} + \nabla(gz) = \nabla(-h + gz)\,, \tag{20}$$

where

$$h = e + \frac{P}{\rho}$$

is the enthalpy of the gas.

With the right-hand side of (17) written as a gradient, the derivation of Kelvin's theorem follows exactly as in section 3. Helmholtz theorem also follows without changes. We have, therefore, the following partial results: For smooth isentropic flows, vortex tubes move with the flow, and their strength remains constant. These results are identical to the ones obtained above for incompressible flows. The vortex stretching mechanism, however, takes a different form, derived in the following section.

8 Vortex Stretching for Compressible Flows

We have seen above that both Kelvin and Helmholtz' theorems remain valid for smooth isentropic compressible flows. Hence vortex tubes move with the flow, and their strength does not change in time. This tells us that, if the cross-section of a vortex tube should shrink, then the mean vorticity across this cross-section should increase proportionally. For incompressible flows, volume conservation implies that the transversal shrinking of a tube is proportional to its longitudinal stretching, hence the vortex stretching mechanism. However, this proportionality does not hold for compressible flows, which may conserve mass without preserving volume, and the vortex stretching mechanism has to be modified to account for this fact.

Schematically, if we think of a cylindrical tube with length L, cross-sectional area A and density ρ, it is not the volume LA but the mass ρLA which has to be preserved by the flow. Since the tube's strength ωA is a constant, we conclude that ω/ρ has to be proportional to L. Thus it is the vector field ω/ρ, not simply ω, which undergoes stretching. Following the same procedure developed above for incompressible flows, we conclude that we can identify ω/ρ with the element of vortex filament to which it is attached, and that the equation

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho}\right) \cdot \nabla u \tag{21}$$

must hold. The formalization of this derivation follows exactly the same steps as in section 6, with the total volume replaced by the mass, and will therefore be omitted.

9 Ertel's Potential Vorticity

Equation (21) is a vector identity. We can derive a useful scalar identity from it by introducing any scalar quantity S which is advected by the flow; i.e. it satisfies the equation

$$\frac{dS}{dt} = 0. \tag{22}$$

An example of such quantity is the concentration of a non-reactive pollutant for time intervals small enough that the effects of diffusion are negligible. It follows from (21) and (22) that the quantity

$$q = \frac{\omega \cdot \nabla S}{\rho} \tag{23}$$

is conserved along particle paths; i.e.

$$\frac{dq}{dt} = 0. \tag{24}$$

The quantity q in (23) is the *p*otential vorticity, first introduced by Ertel.

The meaning of (23) and (24) in terms of vortex stretching is simple. The distance between to surfaces of constant S is inversely proportional to $|\nabla S|$. As this distance changes, the projection of ω onto the direction normal to these surfaces has to change proportionally. This is the content of (24).

Although the formal derivation of (24) from (21) and (22) is straightforward, we will give here an alternative derivation. The reason for this is not only to provide (24) with some intuition, but also to show that this equation remains valid *for non isentropic flows*, even though (21) does not. For this statement to hold, the quantity S has to be such that, together with either P or ρ, it completely determines the thermodynamical state of the fluid. For compressible flows, a quantity satisfying this and (22) is the entropy; hence the letter "S" that we assigned to it.

The point of this latter requirement is that, on a surface of constant S, P and ρ are functionally dependent, even though this functional relation depends on the chosen value of S. Therefore, if one considers only contours $C(t)$ lying on such surface, *Kelvin's theorem remains valid*. (Notice that, if $C(t)$ lies initially on a surface of constant S, it will remain there forever, since both C and S are convected by the flow.) Thus, if the area $\Omega(t)$ enclosed by $C(t)$ shrinks in time, the normal component of the vorticity has to amplify proportionally. In symbols,

$$\Gamma = \int_{\Omega(t)} \frac{\omega \cdot \nabla S}{|\nabla S|} d\Omega = \text{const.} \tag{25}$$

Now consider a tube of fluid between two surfaces $S = S_1$ and $S = S_2$ –not a vortex tube, since these do not move with the flow in the non isentropic case, but any mass of fluid that is convected. In order to compute the mass within the tube, we consider cross-sections of the tube by surfaces of constant S, with areas $\Omega(S, t)$. Then the total mass of the tube is

$$M = \int_{S_1}^{S_2} \int_{\Omega(S,t)} \rho \, d\Omega \frac{dS}{|\nabla S|} = \text{const.} \tag{26}$$

Equation (24) follows from (22), (25) and (26): For each particle, ρ and $\omega \cdot \nabla S$ are proportional, with a constant of proportionality independent of time. This semi-formal derivation can be further formalized, of course, much as in section 6.

The conservation of potential vorticity is a powerful constraint in the study of geophysical flows. For these flows, the vorticity vector is the sum of the local vorticity and the vorticity associated with the rotation of the Earth. Slightly different definitions of the potential vorticity are used when the flows are modeled as incompressible with non-uniform density, and in the shallow water approximation. The potential vorticity is also often linearized in various ways, in important geophysical limits such as those of fast rotation and strong stratification.

10 The Deformation Tensor

We have worried so far about vortex stretching, but not about the flow responsible for this stretching. Equations (4) and (21) describe the stretching of vortex filaments. They both derive from the identity

$$\frac{d}{dt} \frac{\partial}{\partial s} r(s,t) = \frac{\partial}{\partial s} r(s,t) \cdot \nabla u \,, \tag{27}$$

where $\partial r/\partial t = u(r(s,t),t)$. Thus filaments will stretch or not depending on their alignment with the eigenvectors of the tensor ∇u. In this section, we shall study this tensor in more detail.

Let us start by decomposing ∇u into its symmetric and anti-symmentric parts:

$$\nabla u = \frac{\nabla u + \nabla u^{\perp}}{2} + \frac{\nabla u - \nabla u^{\perp}}{2}\,, \tag{28}$$

or

$$\begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} = D + \Omega\,, \tag{29}$$

where

$$D = \begin{pmatrix} u_x & \frac{v_x+u_y}{2} & \frac{w_x+u_z}{2} \\ \frac{u_y+v_x}{2} & v_y & \frac{w_y+v_z}{2} \\ \frac{u_z+w_x}{2} & \frac{v_z+w_y}{2} & w_z \end{pmatrix}\,, \tag{30}$$

the symmetric part of ∇u, is the *d*eformation tensor, and

$$\Omega = \begin{pmatrix} 0 & \frac{v_x-u_y}{2} & \frac{w_x-u_z}{2} \\ \frac{u_y-v_x}{2} & 0 & \frac{w_y-v_z}{2} \\ \frac{u_z-w_x}{2} & \frac{v_z-w_y}{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}\,, \tag{31}$$

the anti-symmetric part of ∇u, which can be written in terms of the three components of the vorticity vector, represents local rotation. The product of this latter matrix and the vorticity vector cancels, so equations (4) and (21) take the simpler forms

$$\frac{d\omega}{dt} = D\omega \tag{32}$$

and

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right) = D\left(\frac{\omega}{\rho}\right). \tag{33}$$

Since the tensor D is symmetric, it has three real eigenvalues with corresponding orthogonal eigenvectors. The vectors ω and ω/ρ amplify maximally when they are oriented along the direction of the eigenvector of D with largest positive eigenvalue; this is the direction of maximal stretching of the flow. For incompressible flows, at least one of the eigenvalues of D is always positive, since the trace of D is zero. This is not necessarily the case for compressible flows, where the fluid may be experiencing a local compression, and all three eigenvalues may be negative. However, even in this case the component of the vorticity along the direction of the eigenvector corresponding to the least negative eigenvalue will be amplifying, even though ω/ρ will not. This follows from the identity

$$\frac{d}{dt}\omega = \rho\frac{d}{dt}\left(\frac{\omega}{\rho}\right) + \left(\frac{\omega}{\rho}\right)\frac{d\rho}{dt} = \rho(D - (\nabla\cdot u)\,\mathrm{I})\left(\frac{\omega}{\rho}\right) \tag{34}$$

and the fact that the trace of D is precisely given by $\nabla \cdot u$.