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Math & Physics of the Atmosphere
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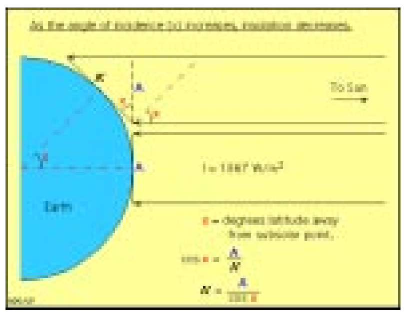
# Radiative and Conductive Heating of the Earth

The goal is to find the temperature distribution as a function of depth and time at a fixed point on the earth, assuming no atmosphere and external heating only due to the sun.

## Radiation

The solar constant measures the radiation emitted by the sun per unit area incident on a plane a distance of 1 astronomical unit from the sun. This constant is equal to $I_o = 1367$ W/m$^2$.

Wherever the sun is directly overhead, the radiation/area is given by the solar constant. However, when the sun is at an angle, the radiation/area is less, since the same radiation covers a larger area. See the figure below (taken with gratitude – but without explicit permission – from the webpage *http://www.ldeo.columbia.edu/edu/dees/ees/climate/slides/insolation_adg.gif*):




We will assume in the following that the sun is directly overhead the equator. Thus, the radiation at a point on earth facing the sun directly and at latitude $\alpha$ is given by

$$I(\alpha) = I_o \cos\alpha$$

If the point is an angle $\theta$ from the side facing the sun, for $\theta$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the insolation is

$$I(\alpha, \theta) = I_o \cos\alpha \, \cos\theta$$

For $\theta$ between $-\pi$ and $-\frac{\pi}{2}$ or between $\frac{\pi}{2}$ and $\pi$, the insolation is 0.

The total solar radiation arriving at the earth then is as follows, where $S$ is the surface of the semisphere receiving radiation and $R = 6370$ km is the earth's radius:

$$\begin{aligned} I_{total} &= \int_S I(\alpha,\theta)\, dA \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I(\alpha,\theta)\, R^2 \cos\alpha \, d\alpha \, d\theta \\ &= R^2 I_o \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\alpha \, d\alpha \, d\theta \end{aligned}$$

$$
\begin{aligned}
&= R^2 I_o \left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{2}\alpha + \frac{1}{4}\sin 2\alpha\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= R^2 I_o \,(2)\,(\frac{\pi}{2}) \\
&= \pi R^2 I_o
\end{aligned}
$$

30% of incoming solar radiation is reflected directly back into space, so that the absorbed radiation at a location $(\alpha, \theta)$ is

$$I_a(\alpha,\theta) = 0.7\, I_o \cos\alpha\, \cos\theta$$

The earth is also rotating once every 86400 seconds. So if we stand at a fixed location on earth, the insolation there will be 0 for half the day and vary from $I_a(\alpha, -\frac{\pi}{2})$ to $I_a(\alpha, \frac{\pi}{2})$ within the other 12 hours. For convenience, we will let $t = 0$ at midnight, so that daylight occurs between $t = 21600$ and $t = 64800$. Then for $0 \le t \le 86400$,

$$I_a(\alpha, t) = \begin{cases} -0.7\, I_o \cos\alpha\, \cos\left(\frac{2\pi t}{86400}\right) & \text{if } 21600 \le t \le 64800 \\ 0 & \text{else} \end{cases}$$

The earth also radiates itself. If we model it as a blackbody, the outgoing radiation of the earth is

$$I_g(t) = \sigma T(t)^4$$

where $\sigma = 5.67\text{x}10^{-8}$ W/m$^2$K$^4$ is the Stefan-Boltzman constant. The net radiation is hence

$$I_a(\alpha, t) - I_g(t) = \begin{cases} -0.7\, I_o \cos\alpha\, \cos\left(\frac{2\pi t}{86400}\right) - \sigma T(t)^4 & \text{if } 21600 \le t \le 64800 \\ -\sigma T(t)^4 & \text{else} \end{cases}$$

The above is all true for the surface of the earth. To see what happens deeper down, we have to consider conduction.

**Conduction**
Conduction occurs according to the heat equation:

$$c\,\rho\, T_t = \kappa T_{zz}$$

where $c = 4000$ J/kgK is the specific heat (of water for our purposes), $\rho = 1000$ kg/m$^3$ is the density (again of fresh water for our purposes) and $\kappa = 0.5$ W/mK is the conductivity constant.

**Numerics**
*Initial Condition:* $T(0, z) = T_e$. $T_e$ is defined to be such that $T(0, z) = T_e$ results in the average surface temperature being $T_e$. (This is found numerically by running Step 1 with different initial temperatures, until the condition holds.)

*Step 1:* We will advance the surface temperature according to the radiation balance, assuming that the layer heated by the radiation has thickness $\Delta z$. The corresponding pde is

$$c\,\rho\,\Delta z\, T_t|_{z=0} = I_a - I_g$$

Using a forward Euler scheme, we calculate:

$$T_0^{n+1} = T_0^n + \frac{\Delta t}{c\,\rho\,\Delta z}\left[\chi(n\Delta t)\left(-0.7\cdot 1370\cos\alpha\cos\left(\frac{2\pi n\Delta t}{86400}\right)\right) - 5.67\text{x}10^{-8}(T_0^n)^4\right]$$

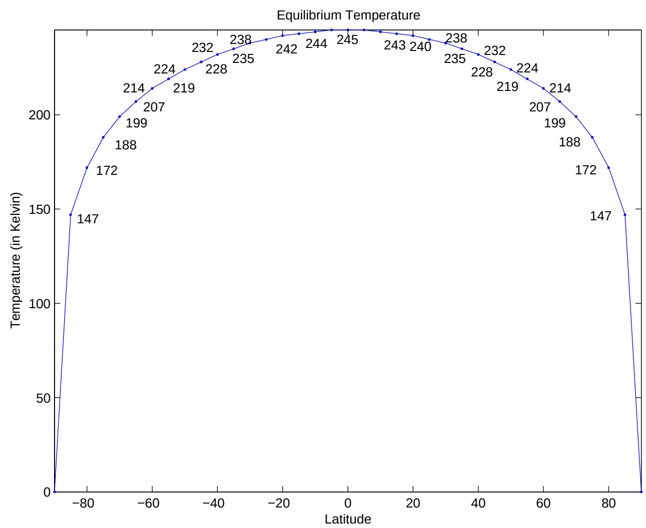
where $\chi$ is the characteristic function on $[21600, 64800]_{\text{mod}\,86400}$.

*Step 2:* Next we evaluate the heat equation in the interior, with the new boundary value for $T$ at the top. For the interior boundary value, we take the neighboring grid point at the new time level. Here, we employ an implicit scheme:

$$T_j^{n+1} = T_j^n + \frac{\kappa\Delta t}{c\,\rho(\Delta z)^2}\left(T_{j+1}^{n+1} - 2\,T_j^{n+1} + T_{j-1}^{n+1}\right)$$

**Numerical Results: Equilibrium Temperature**

The following is a plot of the equilibrium temperatures $T_e$ for various latitudes.




We would like to average this temperature distribution over the volume of the Earth in order to obtain a mean temperature.

For this purpose we divide the terrestrial sphere into slices of thickness $\sin(\alpha + 5^o) - \sin(\alpha)$ and assume each slice to have uniform temperature $T_\alpha = \frac{T(\alpha)+T(\alpha+5^o)}{2}$. The volume of such a slice is given by

$$\begin{aligned} V_\alpha &= \int_{\sin(\alpha)}^{\sin(\alpha+5^o)} \pi(R^2 - y^2)dy \\ &= \pi\,R^3\left[\sin(\alpha+5^o) - \sin(\alpha) - \frac{1}{3}\sin^3(\alpha+5^o) + \frac{1}{3}\sin^3(\alpha)\right] \end{aligned}$$

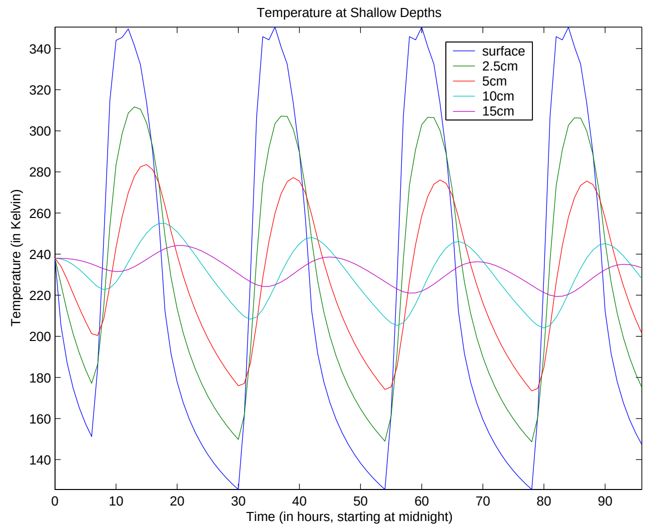
Using the calculated values for $T_e$, we find

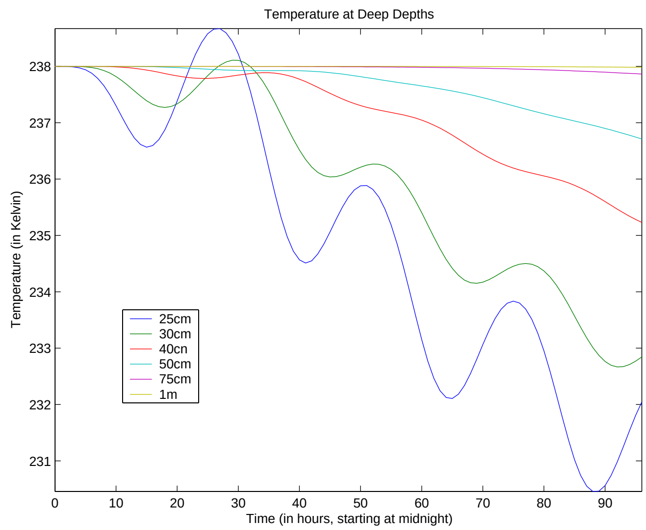
$$\frac{1}{\frac{4}{3}\pi R^3}\sum_\alpha T_\alpha\,V_\alpha = 238\text{ K}$$

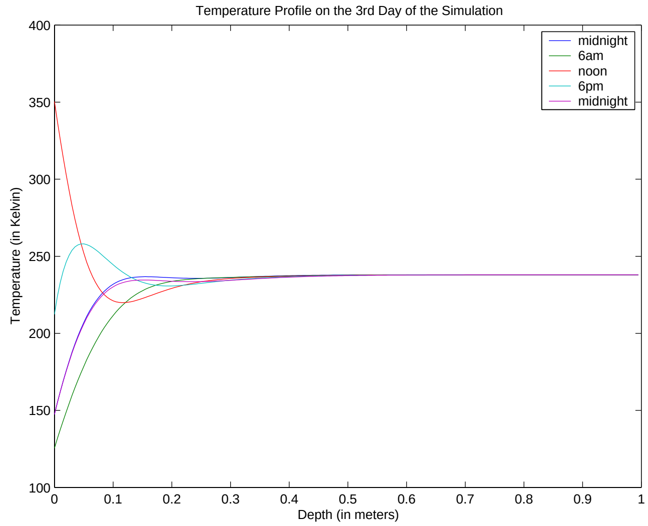
This value is clearly on the cold side (about 34 degrees below freezing). It seems that we do need the atmosphere to account for our warm climate...

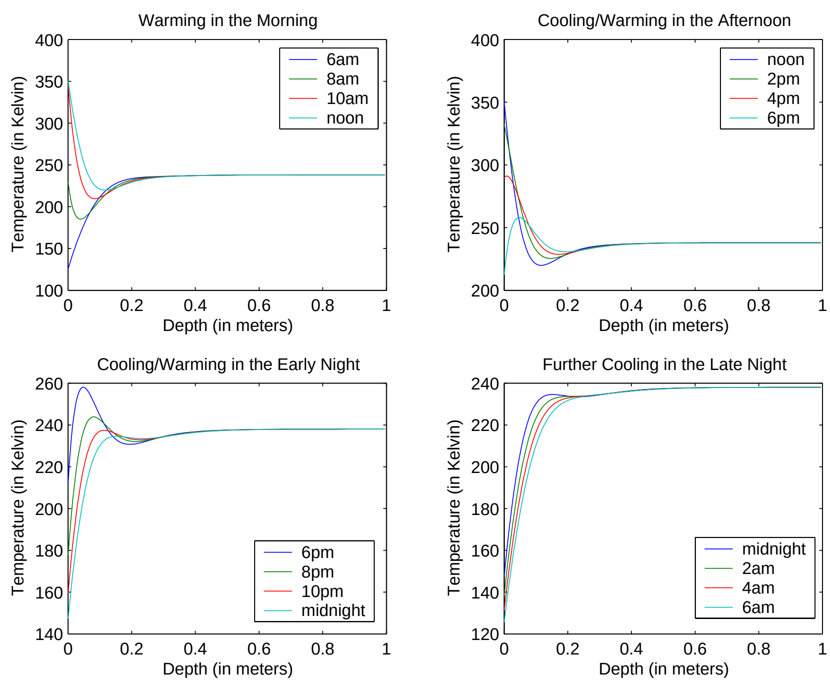
**Numerical Results: Temperature Distributions over Depth**

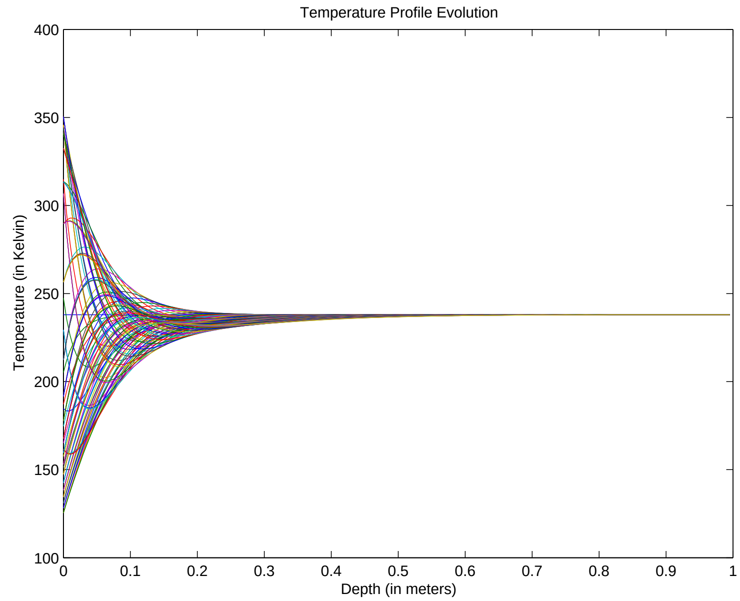
In the following run, we used latitude $\alpha = 30^o$, $\Delta z = 0.5$ cm, maximum depth of 1 m, $\Delta t = 3600$ s, maximum time 96 hours and initial temperature 238 K. These plots illustrate the results:



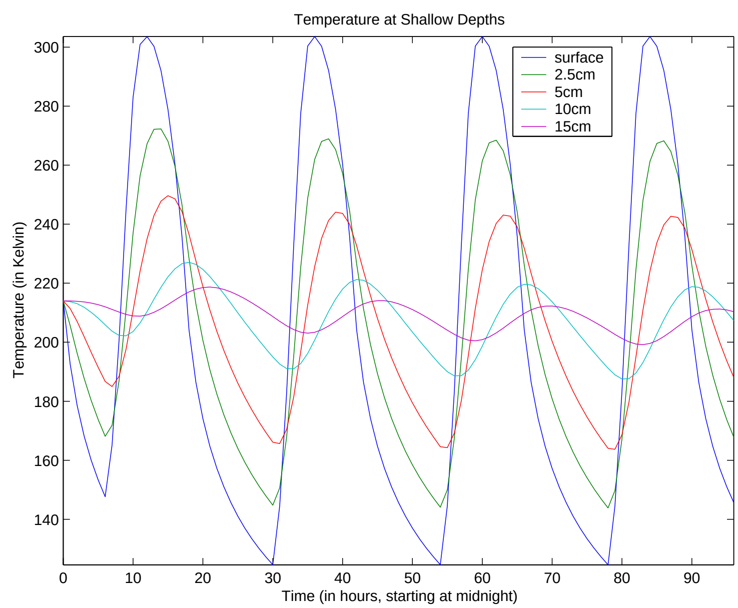

Temperature Profile on the 3rd Day of the Simulation
Temperature (in Kelvin)
Depth (in meters)
midnight
6am
noon
6pm
midnight

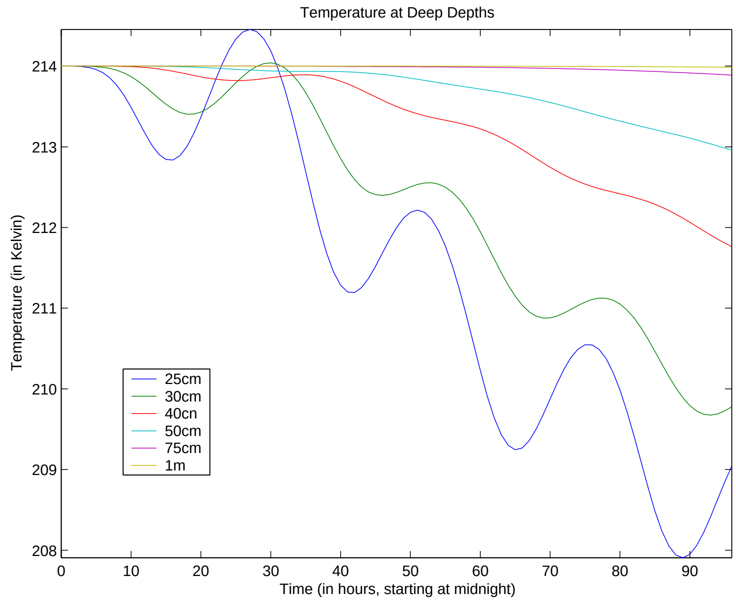

Warming in the Morning
6am
8am
10am
noon
Cooling/Warming in the Afternoon
noon
2pm
4pm
6pm
Cooling/Warming in the Early Night
6pm
8pm
10pm
midnight
Further Cooling in the Late Night
midnight
2am
4am
6am
Temperature (in Kelvin)
Depth (in meters)

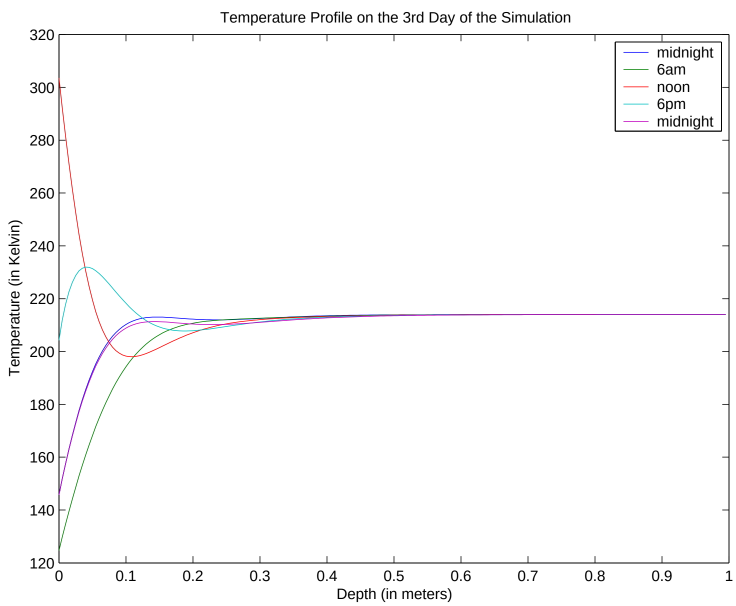


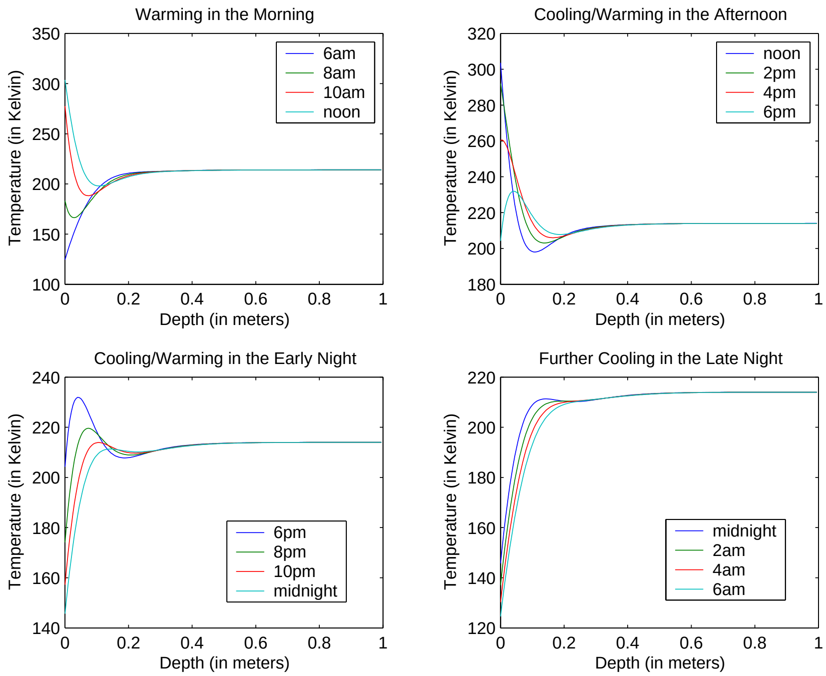

For the next run at latitude $\alpha = 60^o$, we changed the initial condition to 214 K, as this appears to be closer to the relaxation temperature. All other variables were kept the same.

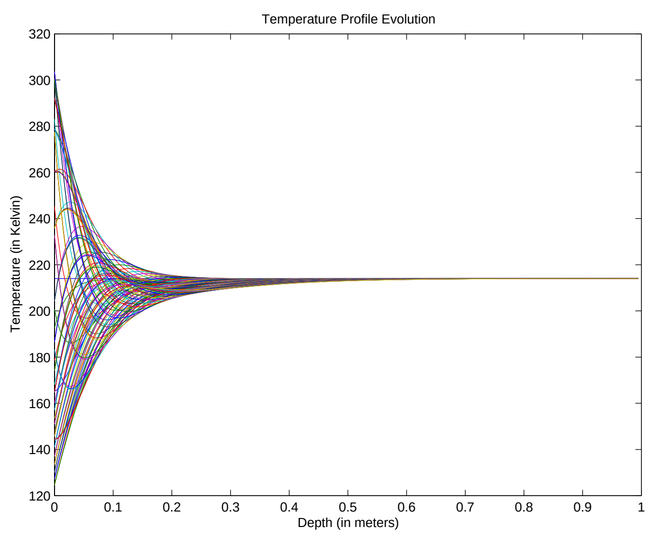
The following results were obtained:



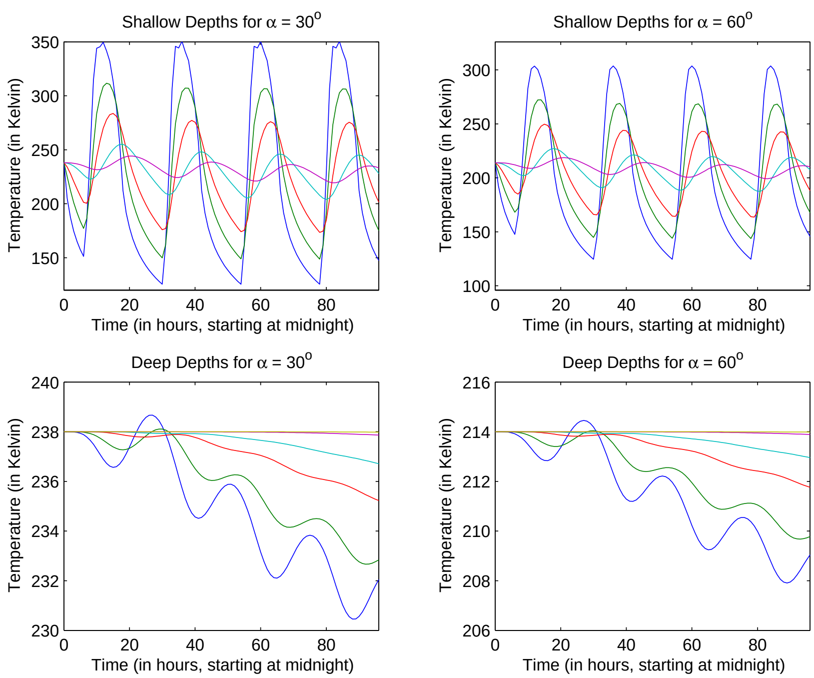

Temperature at Deep Depths
Temperature (in Kelvin)
Time (in hours, starting at midnight)
25cm
30cm
40cn
50cm
75cm
1m
214
213
212
211
210
209
208
0
10
20
30
40
50
60
70
80
90


Temperature Profile on the 3rd Day of the Simulation
Temperature (in Kelvin)
Depth (in meters)
midnight
6am
noon
6pm
midnight
320
300
280
260
240
220
200
180
160
140
120
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1




Note the similarity of the two sets of plots. For the higher latitude, the temperature profiles are more or less a translation of the previous curves to slightly lower temperatures. Worth noting, however, is that some of this similarity is due to a rescaling of the axes. The oscillations are smaller at higher latitudes. While the maximum temperatures reached at each of the two latitudes are quite different, 350 K at $30^o$ and 304 K at $60^o$, the minimum temperatures differ by less than 1 K (125.5 K and 124.6 K, respectively). The cooling mechanism of blackbody radiation tends to be equilibrating, whereas the solar radiation is differentiating various latitudes. The following collection of four plots illustrates these differences.


Shallow Depths for α = 30°
Temperature (in Kelvin)
350
300
250
200
150
0
20
40
60
80
Time (in hours, starting at midnight)
Shallow Depths for α = 60°
Temperature (in Kelvin)
300
250
200
150
100
0
20
40
60
80
Time (in hours, starting at midnight)
Deep Depths for α = 30°
Temperature (in Kelvin)
240
238
236
234
232
230
0
20
40
60
80
Time (in hours, starting at midnight)
Deep Depths for α = 60°
Temperature (in Kelvin)
216
214
212
210
208
206
0
20
40
60
80
Time (in hours, starting at midnight)