Notes for the class: Math and Physics of the Atmosphere, Spring 2004

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1 Radiative models for the Earth

Due to its distance to the Sun, the Earth receives an amount of radiation F_s (the solar constant) given by

$$F_s = 1370 \frac{W}{m^2}$$

(Think of fourteen 100 W light bulbs in a square meter; this should roughly agree with your perception of looking up at the Sun, at midday, a clear summer day.)

1.1 No atmosphere

If the Earth were a flat black body lying in space in the direction facing the Sun, it would absorb all the solar radiation, and radiate an amount given by the Stefan–Boltzman law:

$$F_e = \sigma \, T^4 \, ,$$

where T is its temperature in degrees Kevin (i.e., Celcius, but with the freezing point of water at 273°), and $\sigma = 5.67 \, 10^{-8} \, \frac{W}{m^2 K^4}$. In equilibrium, we would have

$$T = \left(\frac{F_s}{\sigma} \right)^{1/4} \approx 394^\circ$$

somewhat above the boiling point of water. Not a bad starting point as theories go (we could have ended up orders of magnitude away), but obviously in need of some corrections.

First of all, the Earth is not planar but a sphere; hence the total radiation impinging upon it is given by

$$R_s = \pi \, r^2 \, F_s \, ,$$

where $r = 6357 km$ is the Earth's radius. The total radiation emitted by a spherical black–body at uniform temperature T, on the other hand, is

$$R_e = 4 \, \pi \, r^2 \, \sigma \, T^4 \, .$$

Moreover, not all of the radiative energy R_s impinging upon Earth is absorbed: about thirty porcent of it is scattered back into space (i.e., the Earth's *albedo* is about 0.3.) Hence a corrected calculation gives

$$T = \left(\frac{0.7\,F_s}{4\,\sigma}\right)^{1/4} \approx 255^\circ\,,$$

this time sinning on the cold side of things.

At this point, the texts usually bring in the Atmosphere and its warming greenhouse effect. This strikes me as being done a bit early though: how much sense is there in assigning a uniform temperature T to a solid planet which receives solar radiation only from one side? This is the question that I'd like to address next. To this end, we need to combine the effects of radiation and heat conduction, as well as to include the rate of rotation of the Earth.

Consider a solid planet which, for simplicity, we shall assume to have the thermal properties of water, but not its capacity to convect (i.e., carry thermal energy by fluid motion): a density

$$\rho = 1000\frac{kg}{m^3}\,,$$

a heat capacity

$$c = 4000\frac{J}{kg\,K}\,,$$

and a thermal conductivity

$$K = 0.5\frac{W}{m\,K}\,.$$

What are the time scales for thermal equilibration by radiation and by conduction? If we perturb slightly the surface temperature of the planet, its restoring due to radiation will satisfy the law

$$c\,\rho\,h\,T_t = 4\,\sigma\,T^3\,(T_{eq} - T)\,,$$

where h is the depth of the affected surface material. The corresponding time scale for radiation is therefore

$$t_r = \frac{c\,\rho\,h}{4\,\sigma T^3}\,.$$

For vertical conduction, on the other hand, we have

$$c\,\rho\,h\,T_t = K\,\frac{T_{eq} - T}{h}\,,$$

with a corresponding time scale

$$t_c = \frac{c\,\rho\,h^2}{K}\,.$$

The two scales will match when

$$h = \frac{K}{4\,\sigma\,T^3}$$

with corresponding

$$t_c = t_r = \frac{c\,K\,\rho}{16\,\sigma^2 T^6}\,.$$

For $T = 300°$, these yield a depth $h \approx 10cm$, and a time scale $t_c = t_r \approx 12h$.

The small length scale tells us that there will be no significant heat conduction between differentially heated areas of the world. The time scale comparable to a day, on the other hand, speaks of memory, the soil during the day remembering the colder nocturnal temperature, and vis–a–versa (if you try cooler temperatures, the time scale rapidly exceeds a day). Yet it appears that the seasons can be safely neglected; their time scale is much larger than that of radiative restoring.

This suggests writing a model for the evolution of the near surface temperature at latitude α, involving just time and the vertical direction: a heat equation with upper boundary condition given by radiation (variable with the time of the day), and with an asymptotic temperature inside the Earth so as to reach a periodic state. Anybody interested in carryin this through?

1.2 The greenhouse effect

Now we'll add the atmosphere, though a non–convecting one. Away from clouds, the atmosphere is mostly transparent to the solar radiation, but quite opaque to the infraread radiation emanating from the ground. Hence the latter receives not just the direct radiation from the Sun, but also the infraread radiation emited downwards by the atmosphere; this leads to higher ground temperatures than those predicted in the models without an atmosphere.

The simplest model (see the book by Andrews, section 1.3.2), looks at the atmosphere as a whole, letting through a $\tau_s = 0.9$ fraction of the incoming solar radiation (from which the albedo has already been substracted), and a $\tau_g = 0.2$ fraction of that emited by the ground. The remainder is absorbed and re–emited, up and downward in equal shares. It follows that, in equilibrium,

$$\frac{1}{4}\,(1-A)\,F_s = F_a + \tau_g\,F_g\,,$$

and

$$F_g = F_a + \tau_s\,\frac{1}{4}\,(1-A)\,F_s\,,$$

where $A = 0.3$ is the Earth's albedo, the $1/4$ stands for the ratio between the cross–sectional area of the Earth and its surface area, and F_a represents radiation by the atmosphere, both up and downwards. Then

$$F_g = \frac{1}{4}\,(1-A)\,F_s\,\frac{1+\tau_s}{1+\tau_g} = \sigma\,T^4\,,$$

yielding a quite realistic mean ground temperature $T = 286°$.