A NOTE ON GEOMETRIC HEAT FLOWS IN CRITICAL DIMENSIONS

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Abstract. In this note we examine the heat flow for harmonic maps and Yang-Mills equations in dimensions two and four respectively. We show that these two flows are qualitatively similar for degree-2 harmonic maps.

1. Introduction

In this paper we are concerned with a qualitative comparison of phenomena occurring in two different geometric flows: the harmonic map heat flow in two space dimensions and the Yang-Mills heat flow in four space dimensions. These flows are critical in these dimensions, i.e., the Dirichlet energy of maps between Riemannian manifolds, and the Yang-Mills action functional are locally conformally invariant in dimensions two and four respectively.

It has long been established by K.C. Chang, W.-Y. Ding, and R. Ye [CDY] that in 2-dimentions solutions of the equi-variant harmonic map heat flow into $\mathbb{S}^2$ can develop singularities in finite time, while A. Schlatter,M. Struwe, and A.S. Tavildah-Zadeh proved that solutions of the equivariant $\mathbb{SU}(2)$ Yang-Mills heat flow in 4-dimensions remain regular. This different qualitative behavior is surprising given the similarity of the two equi-variant systems: the 2-dimensional harmonic map heat flow

$$\text{(HM)} \qquad \psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\ell^2}{2r^2}\sin 2\psi \stackrel{\text{def}}{=} \psi_{rr} + \frac{1}{r}\psi_r - \frac{\ell^2}{r^2}F(\psi),$$

where ℓ is an integer corresponding to the rotation number of the map and $\psi \in [0,\pi]$; and the 4-dimensional Yang-Mills heat flow

$$\text{(YM)} \qquad \psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{4}{r^2}\psi(1-\psi)(1-\frac{1}{2}\psi) \stackrel{\text{def}}{=} \psi_{rr} + \frac{1}{r}\psi_r - \frac{4}{r^2}G(\psi).$$

Thus if we restrict ψ in the Yang Mills equation to the interval to $[0,2]$ we can interpret (YM) as an equi-varaint harmonic map heat flow, with $\ell = 2$, into a rotationally symmetric surface with a conic singularity at $\psi = 2$. This correspondence will be explained in more details in section 2.

Since the singularity result of K.C. Chang, W.-Y. Ding, and R. Ye was demonstrated for (HM) with $\ell = 1$, the question that we want to answer here is: Is the qualitative difference in behavior between solutions of (HM) and solutions of (YM) due to the different rotation numbers $\ell = 1, 2$ or the presence of the conic singularity in (YM) equation? Our answer is ℓ. Specifically we show that for $\ell = 2$ the Dirichlet problem for the (HM) equation does not blow up as long as $\psi \in [0, 2\pi]$. Moreover we also show that if we replace $\frac{4}{r^2}$ by $\frac{1}{r^2}$ in (YM) then solutions to the modified equation develop singularities.

2. background

Here we give a brief background of the two flows in question and their equi-variant reduction. The interested reader is referred to the survey articles [EL1], [EL2] and the monographs [Jo] and [He] for comprehensive information and references on harmonic maps and the harmonic map heat flow, and to [DK], [St3] and [St4] for the analogous information for the Yang-Mills heat flow.

Date: October 11, 2006.

Harmonic map heat flow. We consider a compact smooth m-dimensional Riemannian manifold (M, g), possibly with nonempty boundary ∂M, and a compact smooth n-dimensional Riemannian manifold (N, h). Although it is sometimes of interest to consider the case that N has a nonempty boundary, here we restrict to the case that N has empty boundary. Via Nash's embedding theorem we can consider N to be isometrically embedded in $\mathbb{R}^k$ for some k. The Sobolev space $H^1(M, N)$ is defined by

$$H^1(M, N) = H^1(M, \mathbb{R}^k) \cap \{u \,\big|\, u(x) \in N \text{ for a.a. } x\}.$$

The *Dirichlet energy* of a map $u \in H^1(M, N)$ to be written as

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2 dvol \ , \tag{2-1}$$

A *harmonic map* is a critical point of (2-1) in $H^1(M, N)$: such a map satisfies the Euler-Lagrange equation

$$\Delta_M u - g^{\alpha\beta} A_u\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right) = 0, \quad i = 1, \ldots, k \ ,$$

where Δ_M is the Laplace-Beltrami operator on M, and A is the second fundamental form of $N \subset \mathbb{R}^k$. The L^2-gradient flow associated to (2-1) is the *harmonic map heat flow*, the system

$$\partial_t u(x, t) = \Delta_M u - g^{\alpha\beta} A_u\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right) \quad \text{on } M \times \mathbb{R}^+.$$

It is standard to refer to t as the time variable, and x as the space variable. One studies the Dirichlet boundary value problem

$$\begin{cases} \partial_t u(x, t) = \Delta_M u - g^{\alpha\beta} A_u\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right) & \text{on } M \times \mathbb{R}^+, \\ u(x, 0) = u_0(x) & \text{on } M, \\ u(x, t)\big|_{\partial M} = \varphi(x) & \text{on } M \times \mathbb{R}^+, \end{cases} \tag{2-2}$$

with a view to producing a harmonic map homotopic to u_0. There are a number of examples of suitable restrictions on the target manifold and/or the initial data which ensure that this behavior will occur. One such restriction is that the the target manifold N have everywhere nonpositive sectional curvature. This situation was considered in [ES], the paper where the harmonic map heat flow (and, indeed, the study of harmonic maps) was introduced.

Although one can consider the harmonic map heat flow for H^1 initial data (see [St1, St2]), we restrict consideration here to the case of smooth u_0. Given smooth initial data, it is standard to show that the problem (2-2) will have a smooth solution for some positive time interval, see [Jo, Chapter 3.2]. In this situation the obstruction to convergence results is the phenomenon of *blow up*: The solution u *blows up* at time $T > 0$ if

$$\limsup_{t \to T^-} ||\nabla u(\cdot, t)||_\infty = \infty \ .$$

The existence of smooth initial data leading to finite-time blow up for the harmonic map heat flow was first demonstrated by Coron–Ghidaglia, [CG], for certain equivariant initial data from $\mathbb{R}^n$ (or S^n) to S^n, for $n \geq 3$; see also [CeD]. The question of finite-time blow up in the critical dimension $n = 2$ was resolved three years later by Chang–Ding–Ye ([CDY]), who demonstrated the existence of initial data from B^2 into S^2 for which finite-time blow up occurs.

We next describe the equivariance we will be considering in the case of the harmonic map heat flow. For maps from Euclidean domains into Euclidean spheres the heat flow in (2-2) reduces to

$$u_t = \Delta u + |\nabla u|^2 u.$$

Given $n \geq 2$ and a function $\psi : [0, \infty) \to \mathbb{R}$, $\psi(0) = 0$, the map

$$u(x) = \Big(\tfrac{x}{|x|}\sin(\psi(|x|), \cos(\psi(|x|))\Big)$$

is an equivariant map from $\mathbb{R}^n$ into S^{n-1}. Writing r for $|x|$, such a map will be harmonic if the function ψ satisfies

$$\psi_{rr} + \frac{n-1}{r}\psi_r - \frac{n-1}{2r^2}\sin 2\psi = 0.$$

For a smooth function $\psi_0 : [0, 1) \to \mathbb{R}$ and any number $\psi_1 \in [0, 2\pi)$ let $\psi : [0, T) \times [0, 1) \to \mathbb{R}$ denote a smooth solution of

$$\begin{cases} \psi_t = \psi_{rr} + \frac{n-1}{r}\psi_r - \frac{n-1}{2r^2}\sin 2\psi \\ \psi(0, r) = \psi_0(r) \\ \psi(t, 0) = 0,\ \psi(t, 1) = \psi_1 \quad t \in [0, T), \end{cases} \tag{2-3}$$

then $u(t, x) = \Big(\frac{x}{|x|}\sin(\psi(t, |x|), \cos(\psi(t, |x|))\Big)$ is a solution to the harmonic map heat flow (2-2) on the unit ball $M = B_1(0)$. We will henceforth restrict attention to this case. From [CD] (cf. [G1, Theorem 4.1]) we have that the equivariance is preserved by the flow (2-2), allowing us to restrict attention to the reduced initial-value problem associated to (2-3).

Note that (2-3) is the L^2- gradient flow associated to the reduced energy functional

$$\mathcal{E}_n = \int_0^1 \left(\psi^2 + \frac{\sin^2\psi}{r^2}\right) r^{n-1}\, dr\,.$$

The equivariance considered can be generalized to *degree-ℓ equivariance*. In order to avoid cumbersome notation, we will only formulate this symmetry in the dimension in which we will be applying it, i.e. $n = 2$. In this case for $\ell \in \mathbb{Z}$, the degree-ℓ equi-variance is defined as follows: let $(r, theta)$ denote polar coordinates on B^2 and (Θ, ϕ) denote standard spherical coordinates (Θ, ϕ) on S^2, ψ being the co-latitude), then ℓ equi-variance is given by

$$\Theta = \ell\theta, \qquad \psi = \psi(t, r).$$

The analogue of (2-3) is the equation

$$\begin{cases} \psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\ell^2}{2r^2}\sin 2\psi \\ \psi(0, r) = \psi_0(r) \\ \psi(t, 0) = 0,\ \psi(t, 1) = \psi_1 \quad t \in [0, T), \end{cases} \tag{2-4}$$

and the analogue of the reduced energy functional $\mathcal{E}_2$ is

$$\mathcal{E}_{2,\ell} = \int_0^1 \left(\psi^2 + \frac{\ell^2\sin^2\psi}{r^2}\right) r\, dr\,.$$

Direct calculation shows that a one-parameter family of finite-energy, static solutions to (2-4) is given by

$$2\arctan\frac{r^\ell}{\lambda} \quad \lambda \in \mathbb{R}^*$$

This family of solutions was used by Chang–Ding–Ye in [CDY] as a basis for constructing a sub-solution of (2-4) on $[0, T) \times [0, \infty)$ and $\ell = 1$ which blows up in finite time. This sub-solution has the form

$$2\arctan\frac{r}{\lambda(t)} + 2\arctan\frac{r}{\mu}$$

for suitable constants μ and ε, and a suitable function λ. With the aid of a standard maximum principle, the authors could then show finite-time blow up for any regular initial data ψ_0 with

$\psi_0 = 0$ and $\psi_0(1) > \pi$. No conclusions were made for $\ell \neq 1$.

Remark. Let N be a smooth, complete, rotationally symmetric compact surface without boundary. This means that N can be identified with a ball of radius $a \in \mathbb{R}^+$ in $\mathbb{R}^2$ equipped with a metric of the form

$$ds^2 = du^2 + g^2(u)d\theta^2 \tag{2-5}$$

where (u, θ) are polar coordinates and $g \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies:

$$g(-x) = -g(x), \quad g(0) = g(a) = 0, \quad g(x) > 0, \text{ for } x \in (0, a), \quad g'(0) = -g'(a) = 1.$$

For such surfaces equi-variant harmonic map heat flows are given by

$$\begin{cases} \psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\ell^2}{r^2} g(\psi)g'(\psi) \\ \psi(0, r) = \psi_0(r) \\ \psi(t, 0) = 0, \, \psi(t, 1) = \psi_1 \quad t \in [0, T), \end{cases} \tag{2-6}$$

Note that if $g'(a) \neq -1$ then the surface N has a conic singularity at $\psi = a$.

Yang-Mills heat flow. We consider a Riemannian manifold M of dimension n, and consider a principal fibre E over M with structure group G, a semi-simple Lie group, and canonical projection π. We denote by $\mathcal{G}$ the Lie algebra of G, and by $[\cdot, \cdot]$ its Poisson bracket. A (smooth) connection on E is a (smooth) map from M into $\operatorname{Ad} E \otimes T^*M$, where $\operatorname{Ad} E$ denotes the adjoint bundle to E. Locally, a connection can be considered as a $\mathcal{G}$-valued 1-form A defined on the coordinate patches U_α of M, $A = A_\mu(x)\, dx^\mu$, with $A_\mu : U_\alpha \to \mathcal{G}$, i.e. for $v \in T_xM$ we have $A(x, u) = A_\mu(x)v^\mu$. A (smooth) gauge transformation is a (smooth) section of the bundle $\operatorname{Aut}(E)$ of automorphisms of E acting on connections by conjugation: a gauge transformation $g : U_\alpha \cap U_\beta \to G$ changing coordinates on fibres in the intersection of the trivializations $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$ acts on A as defined above by

$$g^* A_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g \, .$$

The curvature F_A of a connection A is defined by $F_A = D_A A$, where D_A is the covariant derivative associated with A, and is given locally by the $\mathcal{G}$-valued 2-form $F_{\mu\nu} dx^\mu dx^\nu$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \, .$$

The curvature transforms under g via

$$g^* F_{\mu\nu} = g^{-1} F_{\mu\nu} g \, .$$

This means that the Yang-Mills functional (or Yang-Mills action) $\mathcal{F}$, defined by

$$\mathcal{F}(A) = \int_M F_{\mu\nu} F^{\mu\nu} \, dvol_M \, ,$$

is invariant under gauge transformations. Critical points of $\mathcal{F}$ are *Yang-Mills connections*; they solve the system

$$D^\mu F_{\mu\nu} = 0$$

where $D_\mu = \partial_\mu + [A_\mu, \cdot]$.

One approach to finding Yang-Mills connections is to study the L^2- gradient flow associated with $\mathcal{F}$, the so-called Yang-Mills heat flow, i.e. the initial value problem

$$\begin{cases} \partial_t A_\mu(t, x) = -D^\nu F_{\mu\nu}(t, x) \, , \\ A_\mu(0, x) = A_{0\mu}(x) \end{cases} \tag{2-7}$$

for some suitable initial connection A_0.

As with the harmonic map heat flow, one is interested in the phenomenon of *blow-up* for solutions of (2-7), i.e. the question of whether smooth initial data A_0 can be found such that the solution to (2-7) fails to be smooth after some finite, positive time T. This is known to be the case in dimension $n \geq 5$, as was first shown by Naito in [N], see also [G2]. In subcritical dimensions, i.e. $n = 2, 3$, Råde showed that finite-time blow-up cannot occur. The critical case $n = 4$ is still open for the general problem.

However, Schlatter–Struwe–Tahvildah-Zadeh have shown in [SST], under a symmetry Ansatz which is in many ways the natural analogue of that considered in [CDY], that solutions to (2-7) will not blow up in finite time. A similar Ansatz was used in [CST], [W] and [G2]: in all cases the authors draw on earlier results of [Du] and [I].

To describe the Ansatz, we assume that a symmetry group $\mathcal{S}$ acts on M, the action $\gamma : \mathcal{S} \times M \to M$ being given by $(s, x) \mapsto sx$. We further suppose that γ lifts to an action $\overline{\gamma}$ on the bundle E, i.e. $\overline{\gamma} : \mathcal{S} \times E \to E$, such that: $\gamma \circ \pi = \pi \circ \overline{\gamma}$, i.e.

$$\gamma(s, \pi(z)) = \pi(\overline{\gamma}(s, z)) \qquad \text{for all } s \in \mathcal{S},\ z \in E\,;$$

and such that $\overline{\gamma}$ commutes with the right action of G on E. In this situation we have from [I, Section 2] (cf. [KN, Theorem 11.5]) the existence of a homomorphism $\lambda : \mathcal{S} \to G$ such that, for U a neighbourhood of $x \in M$, on the trivialization $\pi^{-1}(U)$ we have

$$\overline{\gamma}(s, (y, g)) = (sy, \lambda(s)g)) \qquad \text{for all } s \in \mathcal{S},\ y \in U \text{ and } g \in G.$$

The action $\overline{\gamma}$ induces an action on connections. If this action has the effect of a (global) gauge transformation on the local $\mathcal{G}$-valued 1-forms A defined in the introduction, i.e. if

$$A(x, v) = (\lambda(s))^* A(sx, s_* v) \qquad \text{for all } s \in \mathcal{S},\ x \in M \text{ and } v \in T_x M,$$

(where here $s_* : T_x M \to T_{sx} M$ is the push-forward $s_* v = d\gamma(s, v)$), then the connection is called *equivariant* with respect to the $\mathcal{S}$-action γ.

In particular we consider the situation $M = \mathbb{R}^n$, $n \geq 3$, $\mathcal{S} = G = SO(n)$ and E is the trivial bundle $\mathbb{R}^n \times SO(n)$. In this case the homomorphism λ is simply id_G, and leads, as in [Du], to A being given by

$$A_\mu(x) = \frac{\psi(r)}{r^2} \sigma_\mu(x) \tag{2-8}$$

where $r = |x|$, ψ a real-valued function on $[0, \infty)$, and the $\{\sigma_\mu\}_{\mu=1}^n$ are a basis for the Lie-algebra $\mathfrak{so}(n)$, given by

$$\sigma_\mu^{ij}(x) = \delta_\mu^i x^j - \delta_\mu^j x^i \qquad \text{for } 1 \leq i, j, \leq n. \tag{2-9}$$

Connections satisfying (2-8) will be referred to to simply as $SO(n)$-*equivariant connections.*

A straightforward calculation (cf. [Du, Section 3], [W, Chapter 2]) shows that if we have a smooth $SO(n)$-equivariant solution of the Yang-Mills heat flow (2-7) on $[0, T)$, then the system can be rewritten as

$$\begin{cases} \psi_t = \psi_{rr} + \frac{n-3}{r}\psi_r - \frac{2(n-2)}{r^2}\psi(1-\psi)(1-\frac{1}{2}\psi), \\ \psi(0, \cdot) = \psi_0(\cdot) \end{cases} \tag{2-10}$$

Given smooth initial data, the system (2-7) will have a smooth solution for some positive time interval, see [DK, Section 6.3.1] and [St3, Section 4.4]. As was demonstrated by the first author in [G2, Section 3], $SO(n)$-equivariance is preserved by the flow (2-7), allowing one to restrict attention to the reduced initial-value problem (2-10). Note that (2-10) is the L^2- gradient flow associated to the (reduced) energy functional

$$\mathfrak{F}_n(\psi) = \int_0^\infty \left(\psi_r^2 + \frac{2(n-2)}{r^2}\psi^2(1-\frac{1}{2}\psi)^2\right) r^{n-3}\, dr\,. \tag{2-11}$$

Similarly to the case for harmonic maps, we concentrate on the critical dimension $n = 4$. In this case equation(2-10) reduces to

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{4}{r^2}\psi(1-\psi)(1-\frac{1}{2}\psi) \tag{2-12}$$

which has a one-parameter family of finite-energy, static solutions, namely

$$\frac{2r^2}{r^2+\lambda^2} \quad \lambda \in \mathbb{R}\,.$$

This family was used by Schlatter–Struwe–Tavildah-Zadeh in [SST] as a basis for defining a family of supersolutions of (2-10), which in turn were used to to obtain global existence results for (2-7) in the equivariant setting in this dimension. We mention that the static solutions can also be used to define a family of self-similar subsolutions in dimension $n \geq 5$, allowing one to show blow up for suitable initial data in such dimensions; see [G2].

Remark. If we restrict $\psi \in [0, 2]$ in equation (2-12) then we can reinterpret the equation as an equation for a degree-2 equi-variant harmonic map heat flow into a singular surface

$$(N,\, ds^2) = \left([0,2]\times\mathbb{S}^1,\, du^2 + u^2(1-\frac{1}{2}u^2)^2 d\theta^2\right).$$

Comparing the equi-variant equations. Recapping these calculations, the equations for 2-dimensional equi-variant harmonic map heat flow into a compact rotationally symmetric surface (2-5) are given by

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\ell^2}{r^2} g(\psi)g'(\psi); \tag{2-13}$$

while the 4-dimensional equi-variant Yang Mills equation is

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{4}{r^2}\psi(1-\psi)(1-\frac{1}{2}\psi).$$

This later equation can be considered as a member of a family of equations

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\ell^2}{r^2}\psi(1-\psi)(1-\frac{1}{2}\psi) \tag{2-14}$$

which can be interpreted as degree-ℓ equi-variant harmonic maps heat flow into a surface with a conic singularity: $N = [0, 2] \times \mathbb{S}^1$, $ds^2 = du^2 + u^2(1 - \frac{1}{2}u^2)^2 d\theta^2$.

Remark. Direct calculation shows that a one-parameter family of finite-energy, static solutions to (2-14) is given by

$$\frac{2r^\ell}{r^\ell+\lambda^\ell} \quad \lambda \in \mathbb{R}\,.$$

3. The comparisons

At first glance the contrasting results of [CDY] and [SST], i.e. blow up for the harmonic map heat flow for suitable initial data, no blow up for the Yang-Mills heat flow, are surprising. However in reality they are not. In comparing the two flows a natural analogue of the 4-dimensional Yang-Mills heat flow is the degree-2 equi-variant harmonic map heat flow given by (2-4), i.e., we should be comparing equations (2-13) and (2-14) for the same value of ℓ.

We first motivate these claims by studying the nonlinear terms in somewhat more detail. The results are presented in Theorem 3.3 and Theorem 3.4. Rougly speaking, the first of these shows that the degree 2 equivariant harmonic map heat flow does not blow up in finite time, Thus solutions in this case behave more equivariant solutions of the Yang-Mills flow than the degree-1 equivariant harmonic map heat flow. The second result shows finite-time blow up for the equation (2-14) for $\ell = 1$ making it qualitatively closer to the equivariant harmonic map heat flow than to the equivariant Yang-Mills flow.

We record here the maximum principles we will be applying. The proof of Lemma 3.1 follows exactly the same lines as [G1, Lemma 4.3], and the proof of Lemma 3.2 is contained in [G2, Lemma 4.1, Lemma 4.2] (the arguments given there are also valid in dimension $n = 4$)

Lemma 3.1. *Let ψ be a solution of the equi-variant harmonic map heat flow* (2-4) *on $(0,T)\times[0,1]$, and let ξ be a supersolution of the same equation on $(0,T)\times[0,1]$, i.e.,*

$$\xi_t \geq \xi_{rr} + \frac{1}{r}\xi_r - \frac{\ell^2}{2r^2}\sin 2\xi.$$

Suppose that $\psi(0,0) = \xi(0,0) = 0$, and $\psi(0,r) < \xi(0,r)$ for all $r \in (0,1]$. Then $\psi(t,r) < \xi(t,r)$ for all $t \in (0,T)$, $r \in (0,1]$.

Lemma 3.2. *Let ψ be a solution of* (2-12) *for $\ell = 1$ on $(0,T)\times[0,1]$, with $\psi(0,0) = 0$ and $\psi(0,1) = 2$. Let η be a subsolution of the equation satisfied by ψ, i.e.,*

$$\eta_t \leq \eta_{rr} + \frac{1}{r}\eta_r - \frac{4}{r^2}\eta(1-\eta)(1-\frac{1}{2}\eta).$$

Suppose further that $\eta(0,0) = 0$ and $\eta(0,r) < \psi(0,r)$ for all $r \in (0,1]$. Then $\eta(t,r) < \psi(t,r)$ for all $t \in (0,T)$, $r \in (0,1]$.

We are now in a position to state and prove our theorems.

Theorem 3.3. *Let ℓ be a fixed integer, $\ell \geq 2$. Let u_0 be a regular, finite-energy degree-ℓ equivariant map from B^2 to S^2, i.e. using radial coordinates on B^2 and spherical coordinates on S^2, u_0 can be written as*

$$u : (r,\theta) \mapsto (\ell\theta, \psi_0(r)).$$

Suppose that $\psi_0(0) = 0$ and $||\psi_0||_\infty < 2\pi$. Then the harmonic map heat flow with initial data u_0 and boundary data $(t,1,\theta) \mapsto (m\theta, \psi_0(1))$ remains regular for all time, i.e. finite-time blow up will not occur.

Proof: Consider the reduced equation

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\ell^2}{2r^2}\sin 2\psi \stackrel{\text{def}}{=} \mathcal{H}(\psi),$$

with the initial condition $\psi(0,\cdot) = \psi_0(\cdot)$ and boundary condition $\psi(t,1) = \psi_0(1)$. Standard parabolic theory shows that if finite-time blow up occurs, it needs to occur at the origin; see [LSU, Chapter IV, Theorem 10.1] and cf. [G1, Lemma 4.5].

Suppose that, given $T > 0$, we can find a smooth super-solution ξ as stated in lemma 3.1. Then if $\xi_r(t,0)$ is uniformly bounded for $t \in [0,T]$, the maximum principle (Lemma 3.1) implies that

$\psi_r(t,0)$ is uniformly bounded for $t \in [0,T]$, and hence blow up cannot occur for $t \in [0,T]$.

It thus remains to find a suitable super-solution. To this end we consider the static solution

$$\Phi_c = 2\arctan\frac{r^\ell}{c}.$$

For positive constants K, λ_0, and μ to be determined later, we set $\lambda(t) = \lambda_0 e^{-Kt}$, and define $\xi(t,r) = \Phi_{\lambda(t)} + \Phi_\mu$. Then writing φ for $\Phi_{\lambda(t)}$ and β for Φ_μ we have

$$\begin{aligned}
\mathcal{H}(\xi) &= \mathcal{H}(\varphi) + \mathcal{H}(\beta) - \frac{\ell^2}{2r^2}\Big[\sin 2(\phi+\beta) - \sin 2\varphi - \sin 2\beta\Big] \\
&= \frac{\ell^2}{2r^2}\Big[\sin 2\varphi + \sin 2\beta - \sin 2(\varphi+\beta)\Big] \\
&= \frac{\ell^2}{2r^2}\Big[\sin 2\varphi\,(1-\cos 2\beta) + \sin 2\beta\,(1-\cos 2\varphi)\Big] \\
&= \frac{2\ell^2 \sin\varphi\sin\beta}{r^2}\Big[\cos\varphi\sin\beta + \cos\beta\sin\varphi\Big] \\
&= \frac{2\ell^2}{r^2}\sin(\varphi+\beta)\sin\varphi\sin\beta \\
&\leq \frac{4\ell^2\lambda r^\ell}{r^{2\ell}+\lambda^2}\cdot\frac{2\mu r^{\ell-2}}{r^{2\ell}+\mu^2} \\
&\leq \frac{8\ell^2}{\mu}\cdot\frac{\lambda r^\ell}{r^{2\ell}+\lambda^2},
\end{aligned}$$

the last inequality following from the inequality $\frac{2\mu r^{\ell-2}}{r^{2\ell}+\mu^2} \leq \frac{2}{\mu}$, which is valid because $\ell \geq 2$. Since there holds

$$\xi_t = 2K\cdot\frac{\lambda r^\ell}{r^{2\ell}+\lambda^2},$$

if $K > \frac{4\ell^2}{\mu}$ then $\xi_t \geq \mathcal{H}(\xi)$.

Since u_0 has finite energy, we see from the definition of $\mathcal{E}_{2,\ell}$ that ψ_0 must be $\mathcal{O}(r^\ell)$ as $r \to 0$. Since we also assumed $||\psi_0||_\infty < 2\pi$, this means that we can choose μ and λ_0 to ensure that $\psi(0,r) < \xi(0,r)$ for all $r \in (0,1]$. Applying lemma 3.1 we conclude $\psi(t,r) < \xi(t,r)$ for all $t \in (0,T)$, $r \in (0,1]$, meaning that the solution stays regular at $r = 0$. □

Remark: Note that we cannot rule out blow up at time $t = \infty$ by this method. Indeed, consider u_0 as above, with $\psi_0(1) = \pi$, meaning that the boundary data is constant. Then we must have blow up at $t = \infty$. If not, then we have uniform grounds on the gradient, and so a standard compactness argument leads to a sequence of times $t_k \to \infty$ such that $u(t_k,\cdot)$ converges to a smooth harmonic map with constant boundary data. This map is nonconstant, since $\psi_0(0) = 0$ and $\psi_0(1) = \pi$. However, by a result of Lemaire ([L]; see also [KW]), there are no such maps.

Theorem 3.4. *There exists finite energy regular initial data ψ_0 such that the solution to initial-value problem* (2-14) *with $\ell = 1$, i.e.*

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\psi(\psi-1)(\psi-2)}{2r^2} \stackrel{\text{def}}{=} \mathcal{Y}(\psi),$$

blows up in finite time.

Here finite energy means

$$\mathfrak{F}(\psi) = \int_0^\infty \Big(\psi_r^2 + \frac{1}{r^2}\psi^2(1-\frac{1}{2}\psi)^2\Big) r\,dr < \infty\,.$$

Proof: Following [CDY] we construct a suitable subsolution η which blows up in finite time. To this end we consider constants $T > 0$, $\beta > 1$ and $\mu > 0$ and a function $\lambda : [0, T] \to \mathbb{R}$ all to be determined later. We define the following functions on $[0, \infty)$:

$$\phi(t, r) = \frac{2r}{r + \lambda(t)} \quad \text{and} \quad \Phi(r) = \frac{2r^\beta}{r^\beta + \mu},$$

and set

$$\eta = \phi + \Phi.$$

By the remarks at the end of Section 2, we have, for a fixed t, ϕ and Φ are static solutions of (2-14) corresponding to $\ell = 1$ and β respectively. We then calculate

$$\begin{aligned}
\mathcal{Y}(\eta) &= \frac{1}{2r^2}\Big[\phi(\phi-1)(\phi-2) + \beta\Phi(\Phi-1)(\Phi-2) \\
&\qquad -(\phi+\Phi)(\phi+\Phi-1)(\phi+\Phi-2)\Big] \\
&= \frac{1}{2r^2}\Phi\Big[\beta(\Phi-1)(\Phi-2) + \phi(3-2\phi) - (\phi+\Phi-1)(\phi+\Phi-2)\Big] \\
&= \frac{1}{2r^2}\Phi\Big[(\beta-1)(\Phi-1)(\Phi-2) + 3\phi(2-\phi) - 2\phi\Phi\Big] \\
&> \frac{1}{2r^2}\Phi\Big[2(\beta-1) - (4+3(\beta-1))\Phi + (\beta-1)\Phi^2.
\end{aligned}$$

If we restrict to $r \le 1$ (meaning that $\Phi \le \frac{2}{\mu}$) and fix $\beta \in (1, 2)$, we have

$$\mathcal{Y}(\eta) > \frac{1}{2r^2}\Phi\Big[2(\beta-1) - \frac{14}{\mu}\Big].$$

Selecting μ sufficiently large so that $\frac{14}{\mu} < \beta - 1$ implies

$$\mathcal{Y}(\eta) > \frac{\beta-1}{2r^2}\Phi > \frac{\beta-1}{\mu r^{2-\beta}}. \tag{3-1}$$

Now assume that the solution exists to time T. Let

$$\lambda(t) = \frac{1}{2\mu}T^{\frac{-1}{2-\beta}}(T-t)^{\frac{1}{2-\beta}},$$

then $\lambda' = \frac{-(2\mu)^{\beta-2}}{2-\beta}T^{\frac{-\beta}{2-\beta}}\lambda^{\beta-1}$, and λ is monotone decreasing, with $\lambda(0) = \frac{1}{2\mu}$, $\lambda(T) = 0$. We calculate

$$\eta_t = \phi_t = \frac{-2r\lambda'}{(r+\lambda)^2} = \frac{2(2\mu)^{\beta-2}r\lambda^{\beta-1}}{(2-\beta)T^{\frac{\beta}{2-\beta}}(r+\lambda)^2} \tag{3-2}$$

From (3-1) and (3-2), we see that η will be a subsolution on $(0, T) \times (0, 1)$ if

$$\frac{2(2\mu)^{\beta-2}r\lambda^{\beta-1}}{(2-\beta)T^{\frac{\beta}{2-\beta}}(r+\lambda)^2} \le \frac{\beta-1}{\mu r^{2-\beta}}. \tag{3-3}$$

for all r. Making the change of variable $s = r/\lambda$, this is equivalent to showing

$$\frac{2(2\mu)^{\beta-2}s}{(2-\beta)T^{\frac{\beta}{2-\beta}}(1+s)^2} \le \frac{\beta-1}{\mu s^{2-\beta}},$$

or equivalently

$$\frac{s^{3-\beta}}{(1+s)^2} \leq \frac{1}{2}(\beta-1)(2-\beta)T^{\frac{\beta}{2-\beta}}(2\mu)^{1-\beta},$$

for all $s > 0$. The function on the left takes on a unique maximum (which is a function only of β). The right-hand side can be made arbitrarily large by considering T sufficiently large. Thus η is a subsolution on $(0, T) \times (0, 1)$. By Lemma 3.2, any initial data ψ_0 with $\psi_0(0) = 0$ satisfying $\psi_0(r) > \eta(0, r)$ on $(0, 1]$ must blow up before $t = T$. □

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