A deterministic-control-based approach to motion by curvature

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Abstract

The level-set formulation of motion by mean curvature is a degenerate parabolic equation. We show it can be interpreted as the value function of a deterministic two-person game. More precisely, we give a family of discrete-time, two-person games whose value functions converge in the continuous-time limit to the solution of the motion-by-curvature PDE. For a convex domain, the boundary's "first arrival time" solves a degenerate elliptic equation; this corresponds, in our game-theoretic setting, to a minimum-exit-time problem. For a nonconvex domain the two-person game still makes sense; we draw a connection between its minimum exit time and the evolution of curves with velocity equal to the "positive part of the curvature." These results are unexpected, because the value function of a deterministic control problem is normally the solution of a first-order Hamilton-Jacobi equation. Our situation is different because the usual first-order calculation is singular.

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Contents

1 Introduction

We analyze the continuum limit of a family of two-person games. The games are intuitive and easy to understand; they were introduced 25 years ago in connection with problems from combinatorics. Their continuum limit is interesting due to its geometric character. In the simplest case – when the game is played in a convex domain in $\mathbb{R}^2$ – the continuum limit reduces to motion by curvature. Thus our work gives a new game-theoretic interpretation for this geometric evolution law.

Our deterministic-control-based approach to motion by curvature has precursors. A closely related idea was introduced 10 years ago as a numerical approximation scheme for motion by curvature [CDK, Gu]; more recently, similar approximation schemes have been developed for other geometric flows [C, Pa] and in higher dimensions [CL]. These authors' goals and results were however quite different from ours, as we shall explain in Section 1.1.

Motion by curvature is well-understood. Its usual interpretation involves steepest-descent for the perimeter functional. Our game-theoretic interpretation provides an entirely different viewpoint, parallel to the theory of Hamilton-Jacobi equations. There are in fact two different ways of linking an optimal control problem with the associated Hamilton-Jacobi equation. One, known as a "verification argument," works best when the solution is smooth. The other, involving "viscosity solutions," is more general since it requires no smoothness. The analysis of our continuum limit can be done using either technique. The verification argument proves convergence with a rate, while the viscosity-solution argument proves convergence with no rate. The latter result is weaker, of course; but the argument is more universal, since it uses no information about the smoothness of the limiting curvature flow.

Convexity is preserved under motion by curvature, so a convex boundary shrinks monotonically. Thus when Ω is convex, the curvature flow of $\partial\Omega$ can be described in two equivalent ways: by following the moving boundary as an evolving surface, or by specifying for each $x \in \Omega$ the "arrival time" when the boundary reaches x. In our game-theoretic interpretation, the latter viewpoint is associated with a minimum-exit-time problem.

The minimum-exit-time problem makes sense for nonconvex domains, but its continuum limit is not familiar. We characterize its value function in two different ways. First: its level sets are the images of the boundary as it moves by the "positive curvature flow," i.e. with normal velocity $\kappa_+ = \max\{\kappa, 0\}$. Second: it is the unique viscosity solution of an appropriate boundary-value problem. Our proof of the latter statement requires Ω to be star-shaped.

The preceding paragraphs – and most of this paper – address motion by curvature in two space dimensions. Our methods extend, however, to higher dimensions. If the game is not modified, its continuum limit corresponds to the motion of a hypersurface with normal velocity equal to the maximum principal curvature. There is however a natural modification which achieves the mean curvature flow in any dimension.

Our analysis combines two key ideas. One is the level-set approach to the analysis of motion by curvature and related geometric flows (see [ESp, CGG, Gi1]). The other is the analysis of differential games via dynamic programming and Hamilton-Jacobi equations (see e.g. [ESo, Bardi, BC]). Ten years ago it seemed a happy accident that viscosity solutions – invented for applications in optimal control – were also the right tool for analyzing motion by curvature. Now we see that this is no accident; it is in fact quite natural, since motion by curvature can be viewed as an optimal control problem.

Ours is not the first control-theoretic characterization of motion by curvature. An interpretation involving stochastic control was developed in [BCQ, ST1]. There is a link between our deterministic viewpoint and the stochastic framework; we sketch it at the end of Section 1.1.

Our approach is entirely deterministic. One might wonder how such a thing is possible, since deterministic control problems usually lead to first-order Hamilton-Jacobi equations and the level-set formulation of motion by curvature is a second-order PDE. The answer is explained in Section 1.2. Briefly, the Hamilton-Jacobi equation comes from the principle of dynamic programming via Taylor expansion. For our two-person game the Taylor expansion must be carried to second order, leading to a second-order PDE.

1.1 Getting started

To explain in more detail, let's start with the game. Let Ω be a bounded set in $\mathbb{R}^2$. There are two players, Paul and Carol. Paul starts at a point $x \in \Omega$, and his goal is to reach the boundary. Carol is trying to obstruct him. The rules of the game are simple. At each timestep:

1. Paul chooses a direction, i.e. a unit vector $v \in \mathbb{R}^2$ with $\|v\| = 1$.
2. Carol chooses whether to let Paul's choice stand or reverse it – i.e. she chooses $b = \pm 1$ and replaces v with bv.
3. Paul takes a step of size $\sqrt{2}\varepsilon$, moving from x to $x + \sqrt{2}\varepsilon bv$.

Here ε is a small parameter, fixed throughout the game, and we are interested in the continuum limit $\varepsilon \to 0$.

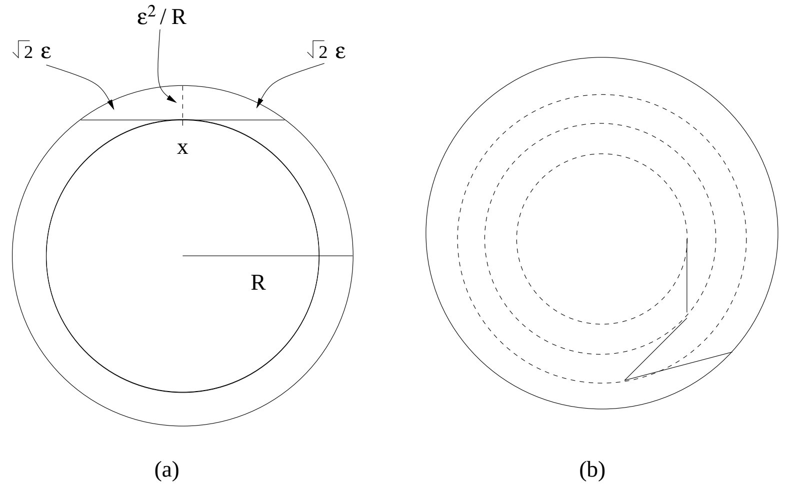

Figure 1: *(a) If Paul starts close enough to the boundary, and he chooses v in the tangent direction, then he exits in one step no matter what Carol does. (b) If Paul starts farther from the boundary, his optimal strategy is to choose v at each stage tangent to the associated circle.*

Can Paul reach the boundary? Yes indeed. The explanation – and the optimal strategy – are easily found using the method of dynamic programming. The key observation is that if Ω is a circle of radius R, then Paul can exit in a single step if his initial position satisfies $|x|^2 + 2\varepsilon^2 > R^2$, in other words if $|x| > R - \Delta R$ with $\Delta R \approx \varepsilon^2/R$. He has only to choose v *tangent* to the circle; Carol cannot stop him, since he exits whether she reverses him or not (see Figure 1a). For initial positions x lying farther from the boundary, we can find the minimum exit time – and the optimal strategies – by repeating this calculation as many times as necessary. For example, starting from the innermost circle shown in Figure 1b, Paul can exit in three steps. He should choose v at each step tangent to the circle on which he sits. (Notice that the optimal paths are not unique: Paul has two equally valid choices of direction at each timestep; moreover the one he actually uses is determined by Carol's whim.)

The figure is convincing, and the argument is local. So it is intuitively clear that for *any* convex domain in the plane, as $\varepsilon \to 0$, the sets from which Paul can exit in a fixed number of steps converge (after an appropriate scaling in time) to the trajectory of $\partial\Omega$ as it evolves under the curvature flow. The main goal of the present paper is to prove this statement and generalize it.

How many steps does Paul need to exit? A convex domain shrinks to a point under motion by curvature [GH, Gr]. Since the area changes at constant rate 2π, its disappearance time T is exactly $|\Omega|/2\pi$. Now, the point x_* to which $\partial\Omega$ shrinks is the location from which Paul needs the most steps to exit. Our results show that starting from x_* he needs approximately T/ε^2 steps to exit.

We digress to link Paul's exit time with Holditch's theorem, a classical result from plane geometry. For any bounded convex domain Ω, consider the curve

$$\gamma_1 = \text{the midpoints of segments of length } 2\sqrt{2}\varepsilon \text{ whose endpoints lie on } \partial\Omega. \tag{1.1}$$

The region inside it is the set from which Paul needs at least two steps to exit; in other words, it is the set where Paul's scaled exit time (defined by (1.3) below) satisfies $U^\varepsilon \geq 2\varepsilon^2$. Holditch's theorem says that the area inside γ_1 is exactly $|\Omega| - 2\pi\varepsilon^2$. (See [Br] for a charming modern discussion of this result, which dates back to 1858. There is a technical condition, always satisfied if ε is small enough: γ_1 must be a simple closed curve, and the orientation of the segments referred to in (1.1) must vary monotonically as their endpoints move around $\partial\Omega$.) This construction can be repeated: for $k = 1, 2, \ldots$ let

$$\gamma_{k+1} = \text{the midpoints of segments of length } 2\sqrt{2}\varepsilon \text{ whose endpoints lie on } \gamma_k.$$

The region bounded by γ_k is the set from which Paul needs at least $k+1$ steps to exit, i.e. where $U^\varepsilon \geq (k+1)\varepsilon^2$. If Holditch's theorem holds for all these curves (i.e. if the technical condition mentioned above is satisfied for all k with ε held fixed) then the area bounded by γ_k is exactly $|\Omega| - 2\pi\varepsilon^2 k$. The process stops when $k = [|\Omega|/2\pi\varepsilon^2]$, since at the next step the area would go negative. Thus we expect – though we have not proved – that Paul can always exit in at most $[|\Omega|/2\pi\varepsilon^2] + 1$ steps.

The preceding discussion – about Paul's exit time – is limited to convex domains. But the game can also be played in a nonconvex domain. In fact, the nonconvex case is interesting and different, because Paul can only exit at the convex part of $\partial\Omega$. We will focus on this topic in Section 1.4.

We first learned of this game from Joel Spencer, who introduced it in [Sp1] (Game 1). It is a variant of his "pusher-chooser" game (see [Sp2]), which is similar except the game is played in $\mathbb{R}^n$ and the number of steps is exactly equal to n. Paul's ability to exit is related to the question of discrepancy of two-color colorings of sets (see also [Sp3]).

The curves γ_k defined above provide a continuous-space, discrete-time approximation to motion by curvature. The "morphological scheme for mean curvature motion" developed in [CDK] gives precisely the same approximation in the convex case. However our goals and results are quite different from those of [CDK]. Indeed:

(a) Our goal is to study the continuum limit of our two-person game and its generalizations; theirs was to give a semidiscrete approximation to motion by curvature in the plane.

(b) For nonconvex domains, our time-discretization of motion by mean curvature remains variational in character (see Section 1.3) while the one in [CDK] does not.

(c) Our analysis emphasizes the connection with control theory. As we explain in Sections 1.2 and 1.3, the level-set equation is like a Hamilton-Jacobi-Bellman PDE. Being elliptic or parabolic, it has no characteristics. And yet Paul's paths play a role closely analogous to those of characteristics in the first-order HJB setting.

(d) Our methods and results are different from those of [CDK] even when our numerical approximation is the same. In fact, when Ω is strictly convex we prove convergence *with a rate*, using a verification argument. The corresponding analysis in [CDK] uses viscosity-solution methods. It is more general, but gets no convergence rate.

There is a relation between our deterministic, two-person game and the stochastic control viewpoint of [BCQ, ST1]. Those papers (specialized to motion by curvature in the plane) assume Paul follows a controlled diffusion process

$$dy = v(y,t)dw \tag{1.2}$$

where v is a unit vector (chosen by Paul, depending on his current position y), w is Brownian motion, and the initial position is $x \in \Omega$. Paul's goal is to choose v so he reaches $\partial\Omega$ by time $t_*(x)$ with probability one, and to make t_* as small as possible. Remembering that Brownian motion can be approximated by a random walk, (1.2) models the situation in which Carol makes her decisions randomly, by flipping an unbiased coin at each step. Paul's optimal strategy (his choice of v) is the *same* for this stochastic problem as for our deterministic one. In the deterministic setting, choosing an optimal v makes him indifferent to Carol's action; choosing any other direction is worse, because Carol will take advantage of his error. In the stochastic setting Carol is passive – she just flips coins – but if Paul makes the wrong choice of v then Carol takes advantage of his error with probability 1/2. This has the same effect as taking advantage of it systematically, since the stochastic problem requires Paul to arrive at the boundary by time t_* *with probability* 1.

The stochastic and deterministic problems are, according to the preceding discussion, closely related. However the tools required to analyze them are rather different. Our deterministic viewpoint is, we think, more elementary. Our Hamilton-Jacobi-Bellman equations are second order due to Taylor expansion; those in [BCQ, ST1] are second order due to Ito's lemma. The stochastic viewpoint has been applied to surfaces of any codimension [ST1, ST2]. Our deterministic viewpoint can be extended similarly; see Example 5 in Section 1.7 for the case of a one-dimensional curve in $\mathbb{R}^3$.

1.2 The minimum exit time, for convex domains in the plane

Our analysis uses the method of dynamic programming. To explain the main ideas, we focus first on the minimum exit time problem, for a bounded convex domain in the plane. For any $\varepsilon > 0$, consider the *minimum exit time*

$$U^\varepsilon(x) = \begin{cases} \varepsilon^2 k \text{ if Paul needs } k \text{ steps to exit, starting} \\ \text{from } x \text{ and following an optimal strategy.} \end{cases} \tag{1.3}$$

Clearly U^ε satisfies the principle of dynamic programming

$$U^\varepsilon(x) = \min_{\|v\|=1} \max_{b=\pm 1} \{\varepsilon^2 + U^\varepsilon(x + \sqrt{2}\varepsilon b v)\}. \tag{1.4}$$

We shall show that as $\varepsilon \to 0$, U^ε converges to the solution of the PDE

$$\begin{cases} \Delta U - \langle D^2 U \frac{\nabla U}{|\nabla U|}, \frac{\nabla U}{|\nabla U|} \rangle + 1 = 0 & \text{in } \Omega \\ U = 0 & \text{at } \partial\Omega. \end{cases} \tag{1.5}$$

This equation was first studied by Evans and Spruck in [ESp]. Its solution has the property that each level set $U = t$ is the image of $\partial\Omega$ under motion by curvature for time t. To see why, consider neighboring level sets $U = t$ and $U = t + \Delta t$ (see Figure 2). If the normal distance between them is Δx then $|\nabla U| \approx \Delta t / \Delta x$, while the curvature of the level set is $\kappa = -\text{div}(\nabla U / |\nabla U|)$. One verifies by elementary manipulation that the PDE (1.5) is equivalent to

$$|\nabla U| \text{div} \left(\frac{\nabla U}{|\nabla U|} \right) + 1 = 0$$

when $|\nabla U| \neq 0$. Thus the PDE says $\kappa = 1/|\nabla U| \approx \Delta x / \Delta t$, whence $\Delta x \approx \kappa \Delta t$ as asserted.

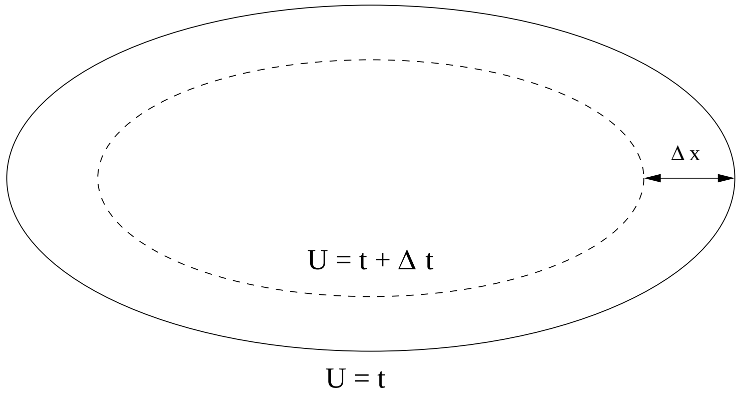

Figure 2: *Level sets of U, the arrival time of the mean curvature flow.*

The analysis of (1.5) in [ESp] uses the framework of viscosity solutions. This is necessary because in its classical form the PDE (1.5) does not make sense where $|\nabla U| = 0$. However there is nothing wrong with the solution. Indeed, for a convex domain, $\partial\Omega$ remains smooth under motion by curvature, and it becomes asymptotically circular as it shrinks to a point [GH, Gr]; using this, we prove in Appendix A that U is C^3 in the entire domain, with $D^2 U(x_*) = -I$ and $D^3 U(x_*) = 0$ at its unique critical point x_*.

The PDE (1.5) is, in essence, the Hamilton-Jacobi-Bellman equation associated with our exit-time problem. To explain, let us derive it heuristically, using the dynamic programming principle (1.4) and Taylor expansion. The former suggests that

$$U(x) \approx \min_{\|v\|=1} \max_{b=\pm 1} \left\{ \varepsilon^2 + U(x + \sqrt{2}\varepsilon b v) \right\}.$$

Expanding U we get

$$U(x) \approx \min_{\|v\|=1} \max_{b=\pm 1} \left\{ \varepsilon^2 + U(x) + \sqrt{2}\varepsilon b v \cdot \nabla U(x) + \varepsilon^2 \langle D^2 U(x) v, v \rangle \right\},$$

which simplifies to

$$0 = 1 + \min_{\|v\|=1} \left\{ \frac{1}{\varepsilon}\sqrt{2}|v \cdot \nabla U(x)| + \langle D^2 U(x)\, v, v \rangle \right\}. \tag{1.6}$$

As $\varepsilon \to 0$ the first term in the minimum requires $v \cdot \nabla U = 0$, in other words $v = \pm \nabla^\perp U / |\nabla U|$, and with this substitution (1.6) becomes

$$0 = 1 + \langle D^2 U \frac{\nabla^\perp U}{|\nabla U|}, \frac{\nabla^\perp U}{|\nabla U|} \rangle. \tag{1.7}$$

Since we are in two space dimensions

$$\Delta U = \langle D^2 U \frac{\nabla U}{|\nabla U|}, \frac{\nabla U}{|\nabla U|} \rangle + \langle D^2 U \frac{\nabla^\perp U}{|\nabla U|}, \frac{\nabla^\perp U}{|\nabla U|} \rangle,$$

so (1.7) can be rewritten as

$$0 = 1 + \Delta U - \langle D^2 U \frac{\nabla U}{|\nabla U|}, \frac{\nabla U}{|\nabla U|} \rangle,$$

which is precisely (1.5).

The preceding calculation, though formal, captures the essence of the matter. It shows, in particular, why the Hamilton-Jacobi-Bellman equation for this game is second rather than first order. The reason is that first-order Taylor expansion does not suffice to characterize U; rather, it tells us only that Paul should choose $v \perp \nabla U$. We need the second-order terms in the Taylor expansion to know how effective this strategy is. Notice that while Carol does not prevent Paul from reaching the boundary, she certainly slows him down. Indeed, Paul's local velocity (step size per time step) is $\sqrt{2}\varepsilon$, but his macroscopic velocity (distance travelled divided by number of time steps) is of order ε^2.

In optimal control, the Hamilton-Jacobi-Bellman equation can be used in two rather different ways. One, known as a "verification argument," uses a solution of the PDE to bound the minimum exit time. The other characterizes the value function of the optimal control problem as the unique viscosity solution of the PDE. The two approaches are complementary, and both are useful for the problem at hand. The viscosity-solution framework is extremely robust, since it requires no information about the PDE solution U; we shall apply it in Sections 4 and 5. When U is smooth enough however the verification argument gives stronger result, namely convergence *with a rate*:

Theorem 1 *Let Ω be a smoothly bounded strictly convex domain in the plane, and let $U(x)$ be the time $\partial\Omega$ arrives at x as it shrinks under motion by curvature, i.e. the solution of (1.5). For $\varepsilon > 0$, let $U^\varepsilon(x)$ be Paul's scaled minimum exit time, defined by (1.3). Then there exists a constant C such that for all $x \in \Omega$*

$$\|U^\varepsilon(x) - U(x)\|_{L^\infty(\overline{\Omega})} \leq C\varepsilon.$$

Moreover C depends only on the C^3 norm of U.

The proof is given in Section 3. The exit problem from a nonconvex domain is discussed in Sections 1.4 and 5.

1.3 Motion by curvature

The curvature flow of $\partial\Omega$ is well-defined even if Ω is not convex. So it should have a game-theoretic interpretation that does not require convexity. We develop such an interpretation here and in Section 4.

The idea is simple: Paul and Carol play the same game as before, but Paul's goal is different. He has an "objective function" u_0 and a "maturity time" T, and his goal is to optimize the value of the objective at maturity. More precisely: his goal is

$$\min u_0(y(T)), \tag{1.8}$$

where $y(s)$ is his piecewise linear path – determined by his choices and Carol's – starting from position x at time t. (His stepsize is $\sqrt{2}\varepsilon$ as before, and each step takes time ε^2.)

This is closely related to our previous discussion. The level sets of u_0 form a nested family of domains in the plane, and Paul's goal is to reach the outermost domain possible by time T. This is different from exiting a specific domain in minimum time – but not very different.

To explain our analysis heuristically, consider Paul's *value function*

$$u^\varepsilon(x,t) = \text{minimal value of } u_0(y(T)), \text{ starting from } x \text{ at time } t. \tag{1.9}$$

It satisfies the dynamic programming principle

$$u^\varepsilon(x,t) = \min_{\|v\|=1} \max_{b=\pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon bv, t + \varepsilon^2). \tag{1.10}$$

We shall show that as $\varepsilon \to 0$, u^ε converges to the solution of

$$\begin{cases} u_t + \Delta u - \langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\rangle = 0 & \text{for } t < T \\ u = u_0 & \text{at } t = T. \end{cases} \tag{1.11}$$

This PDE is familiar from the level-set approach to interface motion. With the time change $\tau = T - t$ it becomes

$$u_\tau - \Delta u + \langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\rangle = 0$$

for $\tau > 0$, with $u = u_0$ at $\tau = 0$. This is equivalent to

$$\frac{u_\tau}{|\nabla u|} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

so the PDE says that each level set of u moves with normal velocity equal to its curvature. In the original time variable t the statement is this: as t decreases from T, each level set $u = c$ describes the curvature flow of the corresponding level set of u_0.

We have been a bit cavalier: it is not obvious how to interpret these PDE's where $|\nabla u| = 0$. The proper framework, developed by Chen, Giga, & Goto [CGG] and Evans & Spruck [ESp], uses the notion of a viscosity solution. This is reviewed in Section 4.

To link the game and the PDE, we argue as in the previous subsection. The dynamic programming principle (1.10) suggests that

$$\begin{aligned} u(x,t) &\approx \min_{\|v\|=1} \max_{b=\pm 1} u(x+\sqrt{2}\varepsilon bv, t+\varepsilon^2) \\ &\approx \min_{\|v\|=1} \max_{b=\pm 1} \left\{ u(x,t) + \sqrt{2}\varepsilon bv \cdot \nabla u + \varepsilon^2 (u_t + \langle D^2 u\, v, v\rangle) \right\} \end{aligned}$$

using Taylor expansion in the second step. This simplifies to

$$0 = \min_{\|v\|=1} \left\{ \frac{1}{\varepsilon}\sqrt{2}|v \cdot \nabla u| + u_t + \langle D^2 u\, v, v\rangle \right\}. \tag{1.12}$$

As $\varepsilon \to 0$ the first term in the minimum requires $v \cdot \nabla u = 0$. Since we are in 2D the remaining terms give precisely (1.11).

We noted earlier that there are two approaches to the rigorous analysis: one via a verification argument, the other via viscosity solutions. The verification argument is behind the proof of Theorem 1, which asserts convergence for the the exit-time problem, with a rate that is linear in ε. The verification argument can also be used in the present setting if $u(x,t)$ is smooth enough. Rather than repeat that argument, however, we prefer to showcase the approach based on viscosity solutions:

Theorem 2 *Consider the game described above, with a continuous "objective function" $u_0 : \mathbb{R}^2 \to \mathbb{R}$ that is constant outside a compact set. Let $u^\varepsilon(x,t)$ be the associated value functions, defined by (1.9). Then the functions u^ε converge as $\varepsilon \to 0$, uniformly on compact sets, to the unique viscosity solution of (1.11).*

The proof is given in Section 4. This theorem is very similar to the convergence result in [CDK, Pa]. The approach in [Pa] is perhaps the most efficient: it uses the fact that any monotone, stable, consistent, scheme is necessarily convergent [BS]. We have chosen to give a different proof, based directly on the optimal control problem. It has the advantage of showing how Paul's paths are like characteristics. In addition, it lays essential groundwork for Section 5.

1.4 The minimum exit time, for nonconvex domains in the plane

Let's return now to the minimum exit time problem. What happens when the domain Ω is nonconvex? Qualitatively, the situation is pretty clear. Paul can still exit – but only from the convex part of $\partial\Omega$. If he starts near the concave part of the boundary he needs many steps to exit, because the convex part of the boundary is far away (see Figure 3). But not *too* many: we'll show in Section 2 that the scaled minimum exit time U^ε, defined by (1.3), is bounded independent of ε.

As in the convex case, our goal is to characterize the limit of U^ε as $\varepsilon \to 0$. In fact we offer two distinct characterizations:

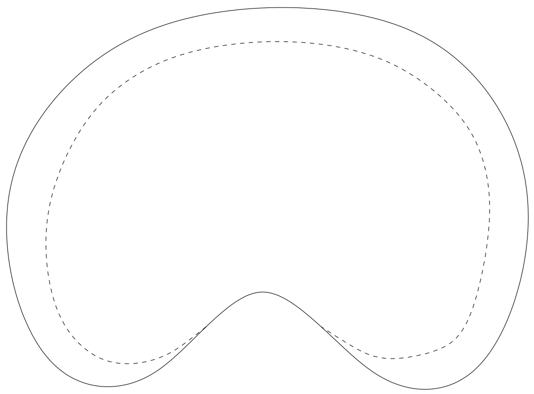

Figure 3: *The dotted line consists of midpoints of segments of length $2\sqrt{2}\varepsilon$ with endpoints on $\partial\Omega$. If Paul starts on or outside it then he can exit in one step.*

(a) it is the unique viscosity solution of the boundary value problem (1.5), interpreting the boundary condition in the viscosity sense; and

(b) its level sets trace the evolution of $\partial\Omega$ under the "positive curvature flow," i.e. the evolution with normal velocity $\kappa_+ = \max\{\kappa, 0\}$.

Our proof of (a) requires Ω to be star-shaped.

The first characterization is more or less expected. It is directly analogous to the situation for the first-order Hamilton-Jacobi equations associated with pursuit-evasion games [BC]. In that context, as in ours, the Dirichlet boundary condition should be imposed only on the part of the boundary from which Paul can exit. By interpreting the boundary condition in the "viscosity sense" – as explained in Section 5 – we assure that it is only imposed on the appropriate part of the boundary.

The proper statement of assertion (a) requires a bit of care, because $\lim_{\varepsilon\to 0} U^\varepsilon$ may not exist, and the relevant viscosity solution can be discontinuous. An example of such behavior is given in Appendix C. Therefore it is natural to consider

$$\overline{U} = \limsup{}^* U^\varepsilon \quad \text{i.e. } \overline{U}(x) = \limsup_{y\to x,\, \varepsilon\to 0} U^\varepsilon(y) \tag{1.13}$$

$$\underline{U} = \liminf{}^* U^\varepsilon \quad \text{i.e. } \underline{U}(x) = \liminf_{y\to x,\, \varepsilon\to 0} U^\varepsilon(x). \tag{1.14}$$

We will prove:

Theorem 3 *Let Ω be a bounded domain in the plane, possibly nonconvex. Let $U^\varepsilon(x)$ be Paul's minimum exit time, defined by (1.3). Then $\overline{U}$, defined by (1.13), is a viscosity subsolution of (1.5); similarly $\underline{U}$, defined by (1.14), is a viscosity supersolution of (1.5).*

Usually a convergence theorem is proved by combining a statement like Theorem 3 with a suitable comparison result. Unfortunately, very little is known about comparison theorems for viscosity solutions of second-order elliptic equations like (1.5) in nonconvex domains. So rather than apply a general comparison result, we must prove one from scratch. Appendix C shows:

Theorem 4 (Barles and Da Lio) *Assume Ω is a bounded, star-shaped domain in $\mathbb{R}^n$. Let u be a viscosity subsolution of (1.5), and let v be a viscosity supersolution. Then $u_* \le v$ and $u \le v^*$, where*

$$u_*(x) = \liminf_{y \to x} u(y), \quad v^*(x) = \limsup_{y \to x} v(y).$$

Taken together, Theorems 3 and 4 show that when Ω is star-shaped, $\overline{U}_* = \underline{U}$ and $\overline{U} = \underline{U}^*$. This characterizes the limiting behavior as the unique-up-to-envelope (possibly discontinuous) viscosity solution of (1.5).

We turn now to the second characterization of the limit. To explain the relevance of the positive curvature flow, consider the modified game in which Paul is no longer required to choose a unit vector – instead, he can choose any v such that $\|v\| \le 1$. For the exit-time problem, we will prove in Section 5.3 that Theorem 3 also holds for the modified game. Thus the $\|v\| = 1$ and $\|v\| \le 1$ versions of the exit-time problem are equivalent: they give the *same* arrival times, at least if Ω is star-shaped. However the time-dependent versions of the two games – with "objective function" u_0, to be minimized at time T – are different. Let us see formally how. Repeating the discussion of Section 1.3 for $\|v\| \le 1$ version of the game, we find that the time-dependent Hamilton-Jacobi-Bellman equation is

$$0 = \min_{\|v\| \le 1} \left\{ \frac{1}{\varepsilon} \sqrt{2} |v \cdot \nabla u| + u_t + \langle D^2 u \, v, v \rangle \right\}.$$

rather than (1.12). As $\varepsilon \to 0$ the first term forces $v \perp \nabla u$. Since

$$\min_{\|v\| \le 1, v \cdot \nabla u = 0} \langle D^2 u(x) v, v \rangle = \left(\langle D^2 u(x) \frac{\nabla^\perp u(x)}{|\nabla u(x)|}, \frac{\nabla^\perp u(x)}{|\nabla u(x)|} \rangle \right)_- , \tag{1.15}$$

using the notation $x_- = \min\{x, 0\}$, the associated PDE is

$$u_t + \left(\langle D^2 u, \frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|} \rangle \right)_- = 0.$$

This is equivalent (in 2D) to

$$u_t + \left(|\nabla u| \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right)_- = 0. \tag{1.16}$$

Since $\kappa = curv(u) = -\mathrm{div}(\nabla u / |\nabla u|)$ is the curvature of a level set of u, the level sets of u solve the "positive curvature flow" backward in time, in other words they flow with

$$\text{normal velocity} = \begin{cases} \kappa & \text{where } \kappa \ge 0, \text{ i.e. the curve is convex} \\ 0 & \text{where } \kappa \le 0, \text{ i.e. the curve is concave.} \end{cases}$$

The existence and uniqueness of this "positive curvature flow" follows from the general framework of [CGG], see also the reference [GG] focusing specifically on curvature flows.

The preceding discussion was formal, but its conclusion is correct:

Theorem 5 *Consider the modified game where Paul's choices are restricted by $\|v\| \le 1$ rather than $\|v\| = 1$. Assume the "objective function" $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and constant outside a compact set. Let $u^\varepsilon(x,t)$ be the associated value functions, defined by (1.9). Then the functions u^ε converge as $\varepsilon \to 0$, uniformly on compact sets, to the unique viscosity solution of (1.16) satisfying $u = u_0$ at $t = T$.*

The point of course is that the minimum-exit-time problem and the positive curvature flow are related. Indeed, we shall show that in the limit $\varepsilon \to 0$, the level sets of Paul's minimum exit time are precisely the images of $\partial\Omega$ as it evolves under the positive curvature flow. The proof uses the underlying game: we show, in essence, that the associated control problems have the same optimal strategy:

Theorem 6 *Let Ω be a bounded domain in the plane (possibly nonconvex), and let $U^\varepsilon(x)$ be Paul's minimum exit time, defined using the modified game in which $\|v\| \le 1$ is permitted. Let $u^\varepsilon(x,t)$ be the value function for the time-dependent version of the modified game, with objective function u_0 and maturity T, and recall from Theorem 5 that the level sets of $u = \lim_{\varepsilon\to 0} u^\varepsilon$ execute the positive-curvature-flow backward in time. Finally, suppose $\Omega = \{u_0 > 0\}$. Then we have*

$$u(x, T - \overline{U}(x)) = u(x, T - \underline{U}(x)) = 0 \tag{1.17}$$

where $\overline{U}$ and $\underline{U}$ are defined in (1.13)–(1.14).

Not much is known about the positive curvature flow of a nonconvex curve in the plane, though it has sometimes been used for image processing, see e.g. [MS]. We conjecture the existence of a free boundary separating the curve into two parts: one strictly convex (moving with normal velocity equal to its curvature), the other strictly concave (and stationary); see Figure 4. Moreover we expect that the concave, stationary part decreases monotonically in size, eventually disappearing – after which the evolution becomes ordinary motion by curvature.

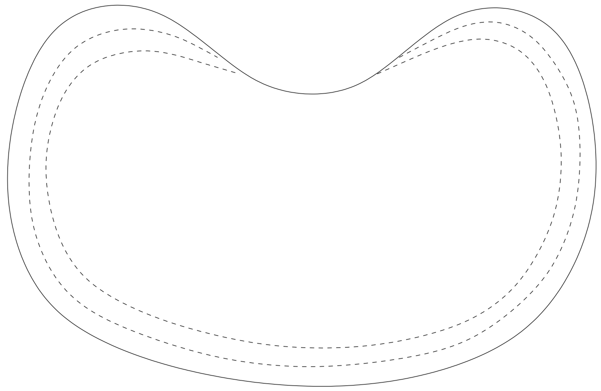

Figure 4: *Schematic of motion by positive curvature for a nonconvex curve in the plane. The solid curve is the locus at a specific time; the dotted curves give the locus at two later times; we expect the "free boundary" separating the convex and concave parts to move along the solid curve.*

What do these conjectures say about the value functions of our games? They suggest that once a part of the curve starts moving it never stops, i.e. that the solution of (1.16) is strictly monotone in time. Thus we ask:

$$\text{is } u_t > 0 \text{ for } x \in \Omega? \tag{1.18}$$

If so, then (1.17) would show $\overline{U} = \underline{U}$, proving convergence of U^ε and continuity of the limit without making use of a comparison theorem. (Such behavior is consistent with the example in Appendix C, since the discontinuity in the example is at the boundary.)

We cannot prove the preceding conjectures, but are able to prove a related result:

Theorem 7 *Let Ω be a domain in the plane with C^2 boundary, and let $\partial\Omega^+$ and $\partial\Omega^-$ be the strictly convex and concave parts of $\partial\Omega$ respectively. Then $\overline{U}$ and $\underline{U}$, defined by (1.13)–(1.14), satisfy*

$$\overline{U}(x) \to 0 \quad \text{as } x \to x_* \in \partial\Omega^+ \tag{1.19}$$

$$\liminf \underline{U}(x) > 0 \quad \text{as } x \to x_* \in \partial\Omega^-. \tag{1.20}$$

This theorem says, in essence, that if Paul starts near a convex part of the boundary he exits quickly; but if he starts near a concave part of the boundary he cannot exit quickly – because he must travel to the convex part of the boundary. The proof of Theorem 7 makes no use of the viscosity framework; rather, it is based directly on the game. In terms of motion by positive curvature (1.20) is a "waiting-time" result: it says that the strictly concave part of $\partial\Omega$ sits still for a nonzero time interval before it begins to move. Note that even though waiting-time results are not known for mean curvature flow, some have been established for the Gauss curvature flow, see for example [CEI].

We are not yet quite done. Our assertion (b) was that the exit times of the *original* game, with $\|v\| = 1$, have level sets given by the positive curvature flow. But Theorem 6 links the positive curvature flow to the exit times of the *modified* game, with $\|v\| \le 1$. To close the loop, we shall show that the two games' exit times yield viscosity sub and supersolutions of the *same* elliptic boundary value problem:

Theorem 8 *Let Ω be a bounded domain in the plane, possibly nonconvex. Let $U^\varepsilon(x)$ be Paul's minimum exit time for the $\|v\| \le 1$ game. Then $\overline{U}$, defined by (1.13), is a viscosity subsolution of (1.5), and $\underline{U}$, defined by (1.14), is a viscosity supersolution of (1.5).*

This theorem closes the loop, provided we have uniqueness for viscosity solutions of (1.5). Such uniqueness is valid for star-shaped domains, as a consequence of Theorem 4.

Theorems 3 through 8 are proved in Section 5, except for Theorem 4 which is proved in Appendix C.

1.5 Higher dimensions

Our discussion has thus far been in the plane. Paul and Carol can play the same game in higher dimensions, but the result is not motion by mean curvature. Rather, it is motion with velocity equal to the largest principal curvature. But the game can easily be modified to give motion by mean curvature in any space dimension. In $\mathbb{R}^3$, for example, the rules are as follows:

(a) Paul chooses two orthogonal unit-length directions, i.e. vectors $v, w \in \mathbb{R}^3$ with $\|v\| = \|w\| = 1$ and $v \perp w$.

(b) Carol chooses whether to let Paul's choices stand or reverse them – i.e. she chooses $b = \pm 1$, $\beta = \pm 1$ and replaces v, w with bv and βw.

(c) Paul takes steps of size $\sqrt{2}\varepsilon$ in each direction, moving from x to $x + \sqrt{2}\varepsilon(bv + \beta w)$.

This game has a simple geometric interpretation, analogous to (1.1): for Paul to exit from a point x in one step, there must exist an "orthonormal cross" of size $2\sqrt{2}\varepsilon$ centered at x whose four endpoints belong to $\mathbb{R}^3 \backslash \Omega$.

To see that this new game works, consider the argument that led to (1.11). The analogue of (1.12) for the modified game is

$$0 = \min_{\substack{\|v\|=\|w\|=1 \\ v \perp w}} \left\{ \frac{1}{\varepsilon} \sqrt{2}(|v \cdot \nabla u| + |w \cdot \nabla u|) + u_t + \langle D^2 u \, v, v \rangle + \langle D^2 u \, w, w \rangle \right\}.$$

As $\varepsilon \to 0$ the coefficient $1/\varepsilon$ forces $v \cdot \nabla u = w \cdot \nabla u = 0$. The terms involving $D^2 u$ give the trace of $D^2 u$ in the plane perpendicular to ∇u. Thus the limiting equation is

$$u_t + \Delta u - \langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle = 0,$$

i.e. level set equation for motion by mean curvature backward in time, valid in any space dimension. It is easy to generalize this procedure for higher dimensions (see Section 6).

This game is very similar to the original one, so it is not surprising that our results extend to it:

Theorem 9 *The natural analogues of Theorems 2 – 8 are valid for this game.*

We justify this assertion in Section 6. Theorem 1 would extend as well if we knew that U (the arrival time of the mean curvature flow) was C^3. However this estimate remains open in higher dimensions. The analogue of Theorem 7 specifies the boundary behavior of U at points where $\partial\Omega$ is strictly concave or strictly convex; it is silent concerning the behavior where the boundary is saddle-shaped.

We shall return to the three-dimensional version of the game in Example 5 of Section 1.7, drawing a connection to the motion of a one-dimensional curve with velocity equal to its curvature vector.

1.6 The inverse game

Here and in Section 1.7 we discuss some natural modifications of the game. They lead, at least formally, to interesting geometric motions.

Consider first the "inverse game." By this we mean the game with the same rules but opposite goals – Paul wants stay in Ω as long as possible, while Carol wants to force him to exit. This reverses the min-max into a max-min. We must stick to the rule $\|v\| = 1$, since otherwise Paul could always avoid exiting trivially by picking $v = 0$. His value function is characterized by

$$U^\varepsilon(x) = \max_{\|v\|=1} \min_{b=\pm 1} U^\varepsilon(x + \sqrt{2}\varepsilon b v). \tag{1.21}$$

The by-now-familiar formal calculation gives the same elliptic equation as before. Thus the exit-time version of the inverse game leads again to motion by positive curvature.

Next, consider the time-dependent version of the "inverse game." Paul's goal is to maximize (rather than minimize) the value of some objective function u_0 at time T. His value function satisfies

$$u^\varepsilon(x,t) = \max_v \min_{b=\pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon b v, t + \varepsilon^2) \tag{1.22}$$

with $u^\varepsilon(x,T) = u_0(x)$. If Paul's choices are restricted to $\|v\| = 1$ then the usual formal calculation leads to the level-set formulation of motion by curvature (1.11). But in the time-dependent setting it makes sense to permit $\|v\| \leq 1$. For this version of the game, arguing as in (1.15), (1.22) leads formally to

$$u_t + \left(\langle D^2 u \frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|}\rangle\right)_+ = 0.$$

In this case the level sets of u execute the "negative curvature flow" backward in time, i.e. the flow with

$$\text{normal velocity} = \begin{cases} \kappa & \text{where } \kappa \leq 0, \text{ i.e. the curve is concave} \\ 0 & \text{where } \kappa \geq 0, \text{ i.e. the curve is convex.} \end{cases}$$

1.7 Other geometric flows

It is natural to ask which other geometric evolutions have game-theoretic interpretations. This question is open, but the following examples show that the class of such evolutions is quite large.

Example 1: Motion by a function times curvature. Suppose Paul's step size is $\sqrt{2}\varepsilon f(x,t,v)$ rather than $\sqrt{2}\varepsilon$. Here f can be any continuous function of position, time, and direction such that $f(x,t,v) = f(x,t,-v)$. Then Paul's value function (for the time-dependent version of the game) satisfies

$$u^\varepsilon(x,t) = \min_{\|v\|=1} \max_{b=\pm 1} u^\varepsilon\left(x + \sqrt{2} b \varepsilon f v, t + \varepsilon^2\right).$$

In the limit $\varepsilon \to 0$ we get

$$u_t + |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) f^2(x,t,\frac{\nabla^\perp u}{|\nabla u|}) = 0.$$

Example 2: Motion by curvature plus a constant. Suppose Paul's motion law is changed as follows: he takes a step of size $\sqrt{2}\varepsilon$ in direction v (possibly reversed by Carol) and also a step of size $\nu\varepsilon^2$ in a direction w which he is free to choose. The associated value function satisfies

$$u^\varepsilon(x,t) = \min_{\|v\|=1} \min_{\|w\|=1} \max_{b=\pm 1} u^\varepsilon\left(x + \sqrt{2} b \varepsilon v + \nu\varepsilon^2 w, t + \varepsilon^2\right).$$

In limit $\varepsilon \to 0$ this gives

$$u_t + |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \nu |\nabla u| = 0.$$

Example 3: Motion by a convex function of curvature. We now explain how to get the continuum law

$$u_t - |\nabla u|\varphi(curv(u)) = 0 \tag{1.23}$$

where $curv(u) = -\mathrm{div}(\nabla u/|\nabla u|)$ is the curvature of the level set of u. The following scheme, an adaptation of one in [Pa], works for any convex function φ which can be represented in the form

$$\varphi(\kappa) = \max_{s\geq 0}\left(\frac{1}{2}\kappa s^2 - |f(s)|\right) \tag{1.24}$$

for some function $f(s)$ such that $f(0) = 0$. The scheme applies, for example, to $\varphi(\kappa) = (\kappa_+)^\gamma$ with $\gamma > 1$, which has the form (1.24) with $f(s) = c_\gamma s^{\frac{2\gamma}{\gamma-1}}$.

Here is the game corresponding to (1.23). At each time step Paul picks an orthonormal frame $v, v^\perp$ and a real number $s \geq 0$; then he moves $\pm\varepsilon s$ in direction v and $\pm\varepsilon^2 f(s)$ in direction $v^\perp$, where Carol chooses both signs. Focusing on the time-dependent version of the game, Paul's value function satisfies

$$u^\varepsilon(x,t) = \min_{\|v\|=1, s\geq 0}\ \max_{b=\pm 1, \beta=\pm 1} u^\varepsilon(x + b\varepsilon s v + \beta\varepsilon^2 f(s)v^\perp, t + \varepsilon^2).$$

The usual formal (Taylor-expansion) argument gives, at order ε, the relation

$$0 = \min_{\|v\|=1, s\geq 0}\ \max_{b=\pm 1} bs\nabla u^\varepsilon(x)\cdot v.$$

This forces $s = 0$ or $\nabla u_\varepsilon(x)\cdot v = 0$; in either case we have $sv = s\frac{\nabla^\perp u_\varepsilon(x)}{|\nabla u_\varepsilon(x)|}$. Proceeding now to the next order ε^2, we get

$$0 = \partial_t u^\varepsilon(x) + \min_{s\geq 0}\left(|f(s)||\nabla u^\varepsilon(x)| + \frac{s^2}{2}\langle D^2 u^\varepsilon(x)\frac{\nabla^\perp u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|}, \frac{\nabla^\perp u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|}\rangle\right).$$

(The choice $s = 0$ is special, since for this s the optimal v is parallel not perpendicular to ∇u^ε; this is why must assume $f(0) = 0$.) Since

$$\langle D^2 u\frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|}\rangle = |\nabla u|\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = -|\nabla u| curv(u)$$

we obtain the limiting PDE

$$u_t + |\nabla u|\min_{s\geq 0}\left(|f(s)| - \frac{s^2}{2}curv(u)\right) = 0.$$

This is equivalent to (1.23) with φ given by (1.24).

It is natural to ask which functions can be represented in the form (1.24) with $f(0) = 0$. The answer is: φ has such a representation if and only if it is lower semicontinuous and convex, and its Fenchel transform (defined by $\varphi^*(k) = \sup_{t\in\mathbb{R}}(tk - \varphi(t))$) satisfies $\varphi^* \geq 0$, $\varphi^*(0) = 0$, and $\varphi^*(t) = +\infty$ for $t < 0$. The proof that these conditions assure the desired representation is easy: any lower semicontinuous convex φ satisfies $\varphi = \varphi^{**}$, i.e.

$$\varphi(k) = \sup_{t\in\mathbb{R}}(tk - \varphi^*(t)). \tag{1.25}$$

If $\varphi^*(t) = +\infty$ for $t < 0$ then the sup is effectively over $t \geq 0$. If in addition $\varphi^* \geq 0$ and $\varphi^*(0) = 0$ then (1.25) gives a representation in the desired form with $|f(s)| = \varphi^*(s^2/2)$. The proof of the converse is only slightly more difficult: if φ has the form (1.24) with $f(0) = 0$ then it is clearly lower semicontinuous and convex. Moreover considering $s = 0$ we see that $\varphi(\kappa) \geq 0$ and $\varphi(0) = 0$, whence $\varphi^*(0) = \max_k\{-\varphi(k)\} = 0$. Now, (1.24) stipulates that $\varphi = g^*$ where

$$g(s) = \begin{cases} +\infty & s < 0 \\ |f(s)| & s \geq 0. \end{cases}$$

Therefore $\varphi^* = g^{**}$ is the largest lower semicontinuous convex function less than or equal to g. It follows easily that $\varphi^* \geq 0$ and $\varphi^*(t) = \infty$ for $t < 0$.

A different but closely related class of examples is obtained by replacing (1.24) by

$$\varphi(\kappa) = \max_{s>0} \left(\frac{1}{2}\kappa s^2 - |f(s)| \right) \tag{1.26}$$

for f such that $|f(s)| \to \infty$ as $s \to 0+$. The game is as above, except that Paul cannot choose $s = 0$. An example of such a φ is

$$\varphi(\kappa) = \begin{cases} -|\kappa|^\gamma & \kappa \leq 0 \\ \infty & \kappa > 0 \end{cases}$$

for $0 < \gamma < 1$, which has the form (1.26) with $f(s) = c_\gamma s^{\frac{2\gamma}{\gamma-1}}$. One verifies that φ has the form (1.26) with $f \to \infty$ as $s \to \infty$ if and only if φ is lower semicontinuous and convex, and its Fenchel transform satisfies $\varphi^* \geq 0$ and $\varphi^*(t) = \infty$ for $t \leq 0$.

Example 4: Motion by a concave function of curvature. This example is parallel to the previous one, except that it uses the "inverse game" in which Paul's goal is to maximize the value of u_0 at time T. It achieves the continuum law

$$u_t - |\nabla u|\varphi(curv(u)) = 0 \tag{1.27}$$

when φ has either the form

$$\varphi(\kappa) = \min_{s\geq 0} \left(\frac{1}{2}s^2\kappa + |f(s)| \right) \tag{1.28}$$

with $f(0) = 0$, or the form

$$\varphi(\kappa) = \min_{s>0} \left(\frac{1}{2}s^2\kappa + |f(s)| \right) \tag{1.29}$$

with $|f(s)| \to \infty$ as $s \to 0$. The framework (1.28) applies to

$$\varphi(\kappa) = \begin{cases} -|\kappa|^\gamma & \kappa \leq 0 \\ 0 & \kappa \geq 0 \end{cases}$$

when $\gamma > 1$, while the framework (1.29) applies to

$$\varphi(\kappa) = \begin{cases} -\infty & \kappa < 0 \\ \kappa^\gamma & \kappa \geq 0 \end{cases}$$

for $0 < \gamma < 1$. In each case the representation uses $f(s) = c_\gamma s^{\frac{2\gamma}{\gamma-1}}$.

The situation is similar to Example 3, so we shall be brief. Let's focus for simplicity on (1.29). The game is like that of Example 3, except that Paul's goal is to maximize rather than minimize his objective at time T. (Thus, he plays the "inverse game" discussed in Section 1.6). His value function satisfies

$$u^\varepsilon(x,t) = \max_{\|v\|=1, s\geq 0} \min_{b=\pm 1, \beta=\pm 1} u^\varepsilon(x + b\varepsilon sv + \beta\varepsilon^2 f(s)v^\perp, t+\varepsilon^2).$$

Arguing as before, we find that Paul's choices should satisfy $sv = s\frac{\nabla^\perp u_\varepsilon(x)}{|\nabla u_\varepsilon(x)|}$, and Taylor expansion to order ε^2 gives

$$0 = \partial_t u^\varepsilon(x) + \max_{s\geq 0}\left(-|f(s)||\nabla u^\varepsilon(x)| + \frac{s^2}{2}\langle D^2 u^\varepsilon(x)\frac{\nabla^\perp u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|}, \frac{\nabla^\perp u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|}\rangle\right).$$

We thus obtain the limiting PDE

$$u_t - |\nabla u| \min_{s\geq 0}\left(|f(s)| + \frac{s^2}{2} curv(u)\right) = 0.$$

This is equivalent to (1.27) with φ given by (1.28).

Example 5: Motion by curvature for one-dimensional curves in $\mathbb{R}^3$. Soner and Touzi have given a stochastic control interpretation of motion by curvature for manifolds of any codimension [ST1, ST2]. Focusing for simplicity on curves in $\mathbb{R}^3$, we now give the analogous deterministic interpretation.

The heart of the matter is the level-set approach to higher-codimension motion by curvature developed by Ambrosio and Soner in [AS]. To model the motion of a curve Γ_0 in $\mathbb{R}^3$ with velocity equal to its curvature vector, they start by choosing a nonnegative, uniformly continuous function u_0 which vanishes exactly on Γ_0. Then they define $u(x,t)$ for $t > 0$ by specifying that $u(x,0) = u_0$ and that each level set of u has normal velocity equal to the *smaller* of its two principal curvatures. (If for example $u_0(x) = \text{dist}(x, \Gamma_0)$ then for $\rho \approx 0$ the set $u_0 = \rho$ is a cylinder of radius ρ about Γ_0. The larger of its principal curvatures is therefore quite large – namely $1/\rho$; however the smaller one remains bounded as $\rho \to 0$, and it approaches the curvature of Γ_0.) The evolution of u is well-defined, and the zero-level-set $\Gamma_t = \{x : u(x,t) = 0\}$ depends only on Γ_0, not on the choice of u_0. Moreover Γ_t agrees with the classical solution of motion by curvature as long as the latter exists. So it is natural to view Γ_t as a weak solution of motion by curvature. (Note however that we cannot rule out "fattening:" once a classical solution ceases to exist the set Γ_t may have positive measure.)

The corresponding game is very simple. Indeed, it is precisely the three-dimensional version of our time-dependent game, using the function u_0 as objective. Paul chooses a unit vector in $\mathbb{R}^3$; Carol accepts or reverses it; then Paul moves distance $\sqrt{2}\varepsilon$ in the resulting direction. His goal is to minimize the value of the objective u_0 at a specified final time. Arguing as in Section 1.3, Paul's value function satisfies (formally, in the limit $\varepsilon \to 0$) the HJB equation

$$u_t + \min_{\|v\|=1,\, v\perp\nabla u} \langle D^2 u\, v, v\rangle = 0 \tag{1.30}$$

for $t < T$, with $u(x,T) = u_0(x)$. The eigenvalues of D^2u on the plane perpendicular to ∇u are precisely $|\nabla u|$ times the principal curvatures of the level set of u. Thus the zero-level-set of u is the Ambrosio-Soner weak solution of motion by curvature, backward in time.

The attentive reader may be puzzled. In Section 1.5 we said the 3D version of the game gave motion with normal velocity equal to the largest principal curvature, yet here we are getting normal velocity equal to the smallest principal curvature. The distinction lies in the convexity or concavity of u_0. In Section 1.5 we assumed Paul's goal was to exit from a region (for the exit-time version of the game) or to get to the outermost of a family of nested regions $u_0 =$ constant by time T (for the time-dependent version of the game). Here Paul's goal is to reach a one-dimensonal target (for an arrival-time version of the game) or to get to the innermost of a family of nested regions (for the time-dependent version).

Additional examples are clearly possible. For example one easily formulates a modified game whose dynamic programming principle is

$$u^\varepsilon(x,t) = \min_{\|v\|=1} \max_{w} \max_{b=\pm 1} u^\varepsilon\left(x + \sqrt{2}b\varepsilon v + \varepsilon^2 w, t + \varepsilon^2\right) - \varepsilon^2 g(w).$$

The associated PDE is

$$u_t + |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + g^*(\nabla u) = 0$$

where g^* is the Legendre transform of g.

1.8 Open questions

Our study raises as many questions as it answers. We collect the main ones here.

1. Let Ω be a bounded convex domain in $\mathbb{R}^n$ with $n \geq 3$, and consider the associated exit time U. For the game described in Section 1.5, U is the arrival time of $\partial\Omega$ as it evolves under motion by mean curvature, and the unique viscosity solution of (1.5). Is it C^3? (If so, then the verification argument in Section 3 works in higher dimensions.)

2. Let Ω be a bounded but nonconvex domain in $\mathbb{R}^2$, and consider the elliptic PDE (1.5) which should determine Paul's exit time. Does it have a comparison principle? Are its viscosity solutions unique? (Appendix C gives an affirmative answer, but only for star-shaped domains.) Also: is the viscosity solution continuous in Ω? (Appendix C shows it need not have a well-defined limit at the boundary.)

3. We argued in Section 1.4 that for nonconvex curves in the plane there should be a free boundary separating the (moving) convex part and the (stationary) concave part (see Figure 4). Is it true that the concave part decreases monotonically in size, and that the curve becomes convex before it shrinks to a point? Can one estimate the velocity of the free boundary? Once a part of the curve starts to move, can it ever stop? (In other words, does the level-set representation have $u_t > 0$ in Ω, c.f. (1.18)?)

4. Does every second-order geometric evolution law have a deterministic game-theoretic interpretation?

5. Does a numerical method based on our game offer any advantages over other more standard approaches to simulating motion by curvature?

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2 Some preliminary results

Section 2.1 gives a more careful definition of the scaled minimum exit time U^ε, and reviews the (easy) proof that it is bounded independent of ε. Section 2.2 proves an elementary but fundamental lemma, which will be used throughout the paper.

2.1 The scaled minimum exit time

We defined U^ε by (1.3). That definition is correct, but somewhat informal. Moreover it does not explain how to evaluate U^ε.

Here is a more constructive viewpoint – essentially, a more careful treatment of the construction implicit in Figures 1 and 3, discussed earlier in connection with Holditch's theorem. It applies to any plane domain, convex or not. If Paul starts at $x \in \mathbb{R}^2\backslash\Omega$ then he is already outside Ω so $U^\varepsilon(x) = 0$. If his starting point x has the property that $x + \varepsilon\sqrt{2}v$ and $x - \varepsilon\sqrt{2}v$ both belong to $\mathbb{R}^2\backslash\Omega$ for some unit vector v, then he can exit in one step and $U^\varepsilon(x) = \varepsilon^2$. Such points x can be characterized as follows: consider all segments of length $2\sqrt{2}\varepsilon$ whose endpoints both lie on $\partial\Omega$ or in $\mathbb{R}^2\backslash\Omega$. The midpoints of these segments define a closed curve γ_1, which partitions $\mathbb{R}^2$ into two regions. The points $x \in \Omega$ from which Paul can exit in one step are those that lie on γ_1 or in the unbounded component of $\mathbb{R}^2\backslash\gamma_1$. Let Ω_1 be the complement of this set in Ω (in other words, Ω_1 consists of those $x \in \Omega$ from which Paul needs two or more steps to exit). Iterating this procedure we find a nested family of subdomains $\Omega_1, \Omega_2, \Omega_3, \ldots$ with $\partial\Omega_j = \gamma_j$ such that Paul needs at least $j+1$ steps to exit from Ω_j. Evidently $U^\varepsilon = (j+1)\varepsilon^2$ on $\Omega_j\backslash\Omega_{j+1}$. We shall show in Lemma 2.1 that if Ω is bounded then Paul can always exit in at most C/ε^2 steps. Therefore Ω_j is empty for j sufficiently large, and the scaled exit time U^ε is uniformly bounded.

The inductive character of this construction is captured by the following alternative definition of U^ε:

Definition 1 *Let Ω be any plane domain, convex or not. If $x \in \mathbb{R}^2\backslash\Omega$ then $U^\varepsilon(x) = 0$. For $x \in \Omega$, $U^\varepsilon(x) = \varepsilon^2$ if there exists a unit-norm vector v such that $x + \sqrt{2}\varepsilon v$ and $x - \sqrt{2}\varepsilon v$ both belong to $\mathbb{R}^2\backslash\Omega$. For any $k \geq 1$, and $x \in \Omega$, $U^\varepsilon(x) = k\varepsilon^2$ if there exists a unit-norm vector v such that $\max(U^\varepsilon(x + \sqrt{2}\varepsilon v), U^\varepsilon(x - \sqrt{2}\varepsilon v)) = (k-1)\varepsilon^2$.*

Actually, we will make little direct use of Definition 1. Rather, our arguments rely mainly on the following characterization:

$$\text{(2.1)} \qquad \begin{cases} U^\varepsilon(x) = \min_{\|v\|=1} \max_{b=\pm 1} U^\varepsilon(x + \sqrt{2}\varepsilon b v) + \varepsilon^2 \text{ for } x \in \Omega \\ U^\varepsilon(x) = 0 \text{ for } x \ \in \mathbb{R}^2\backslash\Omega. \end{cases}$$

This is simply a restatement of (1.4); it follows easily from Definition 1.

The following lemma and its proof are taken from [Sp2].

Lemma 2.1 *For any bounded plane domain Ω, U^ε is uniformly bounded (independent of ε).*

Proof: Let R be the diameter of Ω. Consider any $x \in \Omega$, an arbitrary $b_1 = \pm 1$ and v_1 of norm one, and a path such that

$$\begin{cases} y(0) = x \\ y(\varepsilon^2) = x + \sqrt{2}\varepsilon b_1 v_1 \\ y(k\varepsilon^2) = y((k-1)\varepsilon^2) + \sqrt{2}\varepsilon b_k v_k \end{cases}$$

where for each $k \geq 2$, v_k is taken to be perpendicular to the direction joining $y((k-1)\varepsilon^2)$ to x. Choosing a convenient coordinate system, we may suppose without loss of generality that x is the origin. Whatever the choices of b_k, we have

$$\|y(k\varepsilon^2)\|^2 = \|y((k-1)\varepsilon^2) + \sqrt{2}\varepsilon b_k v_k\|^2 = \|y((k-1)\varepsilon^2)\|^2 + 2\varepsilon^2,$$

since v_k is perpendicular to $y((k-1)\varepsilon^2)$ and $b_k^2\|v_k\|^2 = 1$. It follows that

$$\|y(k\varepsilon^2)\|^2 = 2k\varepsilon^2$$

for all k. In particular, if R is the diameter of Ω we have $y(k\varepsilon^2) \in \mathbb{R}^2\backslash\Omega$ for $2k\varepsilon^2 \geq R^2$.

The argument can be concluded using either the less formal definition (1.3) or else the dynamic programming principle (2.1). To use the former, we observe that the preceding calculation was independent of b_k. So the proposed strategy, though possibly not optimal, assures that Paul needs at most $R^2/(2\varepsilon^2)$ steps to exit. Therefore (1.3) gives $U^\varepsilon \leq R^2/2$.

To use the dynamic programming principle (2.1), assume Carol acts optimally, and let k_* be the first k such that $y(k\varepsilon^2) \in \mathbb{R}^2\backslash\Omega$. We have shown that $k_* \leq R^2/(2\varepsilon^2)$, and by definition $U^\varepsilon(y(k_*\varepsilon^2)) = 0$. An easy inductive argument based on (2.1) gives $U^\varepsilon(y([k_* - j]\varepsilon^2)) \leq j\varepsilon^2$ for $j = 1, 2 \ldots, k_*$. When $j = k_*$ we obtain the same bound as before: $U^\varepsilon(0) \leq k_*\varepsilon^2 \leq R^2/2$. $\square$

We remark that the min in (2.1) is achieved because (holding ε fixed) U^ε is bounded and essentially integer-valued. We will use dynamic programming principles like (2.1) repeatedly in the sequel. The optimal v is always achieved, for the dynamic programming principle associated with any exit-time problem, by the argument just given.

2.2 The fundamental lemma

The crucial step in our formal argument was the passage from (1.6) to (1.7). The following lemma will be used be used repeatedly to make that argument (and others like it) rigorous.

Lemma 2.2 *Let ϕ be a C^3 function on a compact subset of $\mathbb{R}^2$. Then*

(a) for any x and any vector v, we have

$$\max_{b=\pm 1} \phi(x + \sqrt{2}\varepsilon bv) \geq \phi(x) + \varepsilon^2\langle D^2\phi(x)\, v, v\rangle - C\varepsilon^3.$$

(b) for any x such that $\nabla\phi(x) \neq 0$ we have

$$\min_{\|v\|=1} \max_{b=\pm 1} \phi(x + \sqrt{2}\varepsilon bv) \leq \phi(x) + \varepsilon^2\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle + C\varepsilon^3;$$

(c) if $\nabla\phi(x) \neq 0$, we have

$$\min_{\|v\|=1}\max_{b=\pm 1}\phi(x+\sqrt{2}\varepsilon bv) \geq \phi(x) + \varepsilon^2\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle - C(1+\frac{1}{|\nabla\phi(x)|})\varepsilon^3.$$

In each estimate, the constant C depends only on the C^3 norm of ϕ, not on x or ε.

Proof: Part (a) is an immediate consequence of the Taylor expansion

$$\phi(x+\sqrt{2}\varepsilon bv) = \phi(x) + \sqrt{2}\varepsilon bv\cdot\nabla\phi(x) + \varepsilon^2\langle D^2\phi(x)v, v\rangle + O(\varepsilon^3). \tag{2.2}$$

For part (b), we observe that when $v = \nabla^\perp\phi/|\nabla\phi|$, (2.2) gives

$$\max_{b=\pm 1}\phi(x+\sqrt{2}\varepsilon bv) = \phi(x) + \varepsilon^2\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle + O(\varepsilon^3).$$

The minimum of the left hand side over all $\|v\| = 1$ can only be smaller, so (b) is proved.

For (c), we begin by observing the existence of a constant c_1 (depending only on the C^2 norm of ϕ) with the following two properties for all unit vectors v: (i) $\sqrt{2}|\nabla\phi(x)\cdot v| \geq c_1\varepsilon$ implies

$$\sqrt{2}\varepsilon|\nabla\phi(x)\cdot v| + \varepsilon^2\langle D^2\phi(x)v, v\rangle \geq \varepsilon^2\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle;$$

and (ii) $\sqrt{2}|\nabla\phi(x)\cdot v| \leq c_1\varepsilon$ implies

$$\sqrt{2}\varepsilon|\nabla\phi(x)\cdot v| + \varepsilon^2\langle D^2\phi(x)v, v\rangle \geq \varepsilon^2\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle - \frac{c_2\varepsilon^3}{|\nabla\phi(x)|}.$$

Indeed, property (i) holds provided $c_1 \geq 2\max_{\|v\|=1}|\langle D^2\phi(x)v, v\rangle|$. The hypothesis of (ii) on the other hand implies that

$$\left|\langle\frac{\nabla\phi(x)}{|\nabla\phi(x)|}\cdot v\rangle\right| \leq \frac{c_1\varepsilon}{\sqrt{2}|\nabla\phi|}.$$

If $\frac{\varepsilon}{|\nabla\phi(x)|}$ is small, then the angle between v and $\nabla\phi(x)/|\nabla\phi(x)|$ is close to $\pi/2$ and so v is close to $\pm\nabla^\perp\phi(x)/|\nabla\phi(x)|$, more precisely

$$\left|v \pm \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\right| = O\left(\frac{\varepsilon}{|\nabla\phi(x)|}\right).$$

This inequality is also trivially true when $\frac{\varepsilon}{|\nabla\phi(x)|}$ is not small. It follows that

$$\varepsilon^2\langle D^2\phi(x)v, v\rangle = \varepsilon^2\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle + O\left(\frac{\varepsilon^3}{|\nabla\phi(x)|}\right),$$

and this implies property (ii).

With these preliminaries in hand, part (c) of the Lemma is easy. For any unit vector v, maximization over $b = \pm 1$ in the Taylor expansion (2.2) gives

$$\max_{b=\pm 1}\phi(x+\sqrt{2}\varepsilon bv) \geq \phi(x) + \sqrt{2}\varepsilon|v\cdot\nabla\phi(x)| + \varepsilon^2\langle D^2\phi(x)v, v\rangle + O(\varepsilon^3).$$

If $\sqrt{2}|\nabla\phi(x)\cdot v| \geq c_1\varepsilon$ we estimate the right hand side using alternative (i); if the opposite inequality holds we estimate it using alternative (ii); either way, we get

$$\max_{b=\pm 1}\phi(x+\sqrt{2}\varepsilon bv) \geq \phi(x) + \varepsilon^2\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle - C(1+\frac{1}{|\nabla\phi|})\varepsilon^3.$$

Now minimization over v gives the conclusion of part (c). $\square$

3 The minimum exit time, analyzed by a verification argument

This section proves Theorem 1. The analysis is restricted to bounded, convex domains in the plane. We can use a verification argument because the solution of the limiting PDE (1.5) is classical. In fact, we have the following result, which will be proved in Appendix A:

Lemma 3.1 *Let Ω be a smoothly bounded strictly convex domain in the plane, and let $U(x)$ be the time $\partial\Omega$ arrives at x as it shrinks under motion by curvature. Then*

(a) U is C^3 in $\overline{\Omega}$ and solves the degenerate elliptic equation (1.5).

(b) U has just one critical point x_* in Ω. At this point U achieves its maximum and $D^2U(x_*) = -I$.

(c) $D^3U(x_*) = 0$.

Our verification argument has two distinct parts. One gives an upper bound on U^ε by considering a specific, possibly suboptimal strategy. The other gives a lower bound on U^ε by considering the optimal strategy. We present the two parts separately, as Propositions 3.1 and 3.2. The proofs use parts (a) and (b) of Lemma 3.1 but not part (c).

Proposition 3.1 *Let Ω be a smoothly bounded, strictly convex domain in the plane. Then*

$$U^\varepsilon(x) \le U(x) + C\varepsilon \tag{3.1}$$

for all sufficiently small $\varepsilon > 0$ and every $x \in \Omega$. The constant C depends on Ω, but not on x or ε.

Proof: Let us first prove that U^ε is lower semi-continuous. Since U^ε only takes values $k\varepsilon^2$ ($k \in \mathbb{N}$), it is sufficient to prove that for every k, $\Sigma_k = \{x \in \mathbb{R}^2, U^\varepsilon(x) \le k\varepsilon^2\}$ is closed. The case $k = 0$ is obvious. Suppose by induction that Σ_k is closed. Assume that $x_n \in \Sigma_{k+1}$ converges to $x_0 \in \mathbb{R}^2$. Then by Definition 1 there exists v_n with $\|v_n\| = 1$ such that

$$\max(U^\varepsilon(x_n + \sqrt{2}\varepsilon v_n), U^\varepsilon(x_n - \sqrt{2}\varepsilon v_n)) \le \varepsilon^2 k.$$

In other words, $x_n \pm \sqrt{2}\varepsilon v_n \in \Sigma_k$. We may assume that v_n converges to v_0 by taking a subsequence. Since Σ_k is assumed to be closed, we see that $x_0 \pm \sqrt{2}\varepsilon v_0 \in \Sigma_k$. By Definition 1 this implies that $U^\varepsilon(x_0) \le \varepsilon^2(k+1)$ i.e. $x_0 \in \Sigma_{k+1}$ so that Σ_{k+1} is closed, and U^ε is lower semi-continuous.

Let now x_* be the unique critical point (maximum) of U. It is sufficient, by continuity of U and lower semi-continuity of U^ε, to prove (3.1) for $x \ne x_*$. So consider any $x_0 \ne x_*$ in Ω, and let $v_0 = \nabla^\perp U(x_0)/|\nabla U(x_0)|$. From the dynamic programming principle (2.1), we have

$$U^\varepsilon(x_0) \le \max_{b=\pm 1} U^\varepsilon(x_0 + \sqrt{2}\varepsilon b v_0) + \varepsilon^2.$$

Let b_0 achieve the maximum in this relation, and consider $x_1 = x_0 + \sqrt{2}\varepsilon b_0 v_0$. By construction, it satisfies

$$U^\varepsilon(x_0) \le U^\varepsilon(x_1) + \varepsilon^2.$$

In addition, Taylor expansion (2.2) combined with the PDE (1.5) give

$$
\begin{aligned}
U(x_1) &= U(x_0) + \varepsilon^2 \langle D^2U(x_0)\frac{\nabla^\perp U(x_0)}{|\nabla U(x_0)|}, \frac{\nabla^\perp U(x_0)}{|\nabla U(x_0)|}\rangle + O(\varepsilon^3) \\
&= U(x_0) - \varepsilon^2 + O(\varepsilon^3). \qquad (3.2)
\end{aligned}
$$

If ε is small enough, (3.2) gives $U(x_1) < U(x_0)$, which shows in particular that $x_1 \neq x_*$. Therefore we can iterate the preceding construction. This gives a sequence $x_k \in \Omega$ and $b_k = \pm 1$ ($k = 1, 2 \ldots$) with $x_{k+1} = x_k + \sqrt{2}\varepsilon b_k \nabla^\perp U(x_k)/|\nabla U(x_k)|$, such that $U^\varepsilon(x_k) \leq U^\varepsilon(x_{k+1}) + \varepsilon^2$ and $U(x_{k+1}) = U(x_k) - \varepsilon^2 + O(\varepsilon^3)$. Adding these relations as k varies, we have

$$U^\varepsilon(x_0) \leq U^\varepsilon(x_k) + k\varepsilon^2 \qquad (3.3)$$

and

$$U(x_k) = U(x_0) - k(\varepsilon^2 + O(\varepsilon^3)). \qquad (3.4)$$

The iteration stops when the sequence x_k leaves Ω. Now, U is positive and bounded in Ω, whereas (3.4) would force $U(x_k)$ to be negative for $k > U(x_0)/(\varepsilon^2 + O(\varepsilon^3))$. Therefore the sequence terminates in at most $U(x_0)/(\varepsilon^2 + O(\varepsilon^3))$ steps, i.e. there exists $K < U(x_0)/(\varepsilon^2 + O(\varepsilon^3))$ (K depends on ε and x_0) such that $x_K \in \Omega$ but $x_{K+1} = x_K + \sqrt{2}\varepsilon b_K \nabla^\perp U(x_K)/|\nabla U(x_K)|$ lies outside Ω (here the $O(\varepsilon^3)$ is the same function as in (3.4)).

Since x_{K+1} lies outside Ω we have $U^\varepsilon(x_{K+1}) = 0$ and $U^\varepsilon(x_K) = \varepsilon^2$; therefore (3.3) becomes

$$U^\varepsilon(x_0) \leq (K+1)\varepsilon^2.$$

On the other hand, (3.4) with $k = K$ gives

$$U(x_0) = U(x_K) + K(\varepsilon^2 + O(\varepsilon^3)) \geq K\varepsilon^2 + O(\varepsilon)$$

since $K \leq \frac{C}{\varepsilon^2}$ and $U(x_K) > 0$. Subtracting these relations, we conclude that $U^\varepsilon(x_0) - U(x_0) \leq O(\varepsilon)$. This proves the Proposition. □

Proposition 3.2 *Let Ω be a smoothly bounded, strictly convex domain in the plane. Then*

$$U^\varepsilon(x) \geq U(x) - C\varepsilon$$

for all sufficiently small $\varepsilon > 0$ and every $x \in \Omega$. The constant C depends on Ω but not on x or ε.

Proof: Let $x = x_0$ be in Ω. We start again from the characterization (2.1), but this time we choose v_0 to achieve the min in (2.1):

$$U^\varepsilon(x_0) = \max_{b=\pm 1} U^\varepsilon(x_0 + \sqrt{2}\varepsilon b v_0) + \varepsilon^2;$$

in particular, for each $b = \pm 1$ we have

$$U^\varepsilon(x_0) \geq U^\varepsilon(x_0 + \sqrt{2}\varepsilon b v_0) + \varepsilon^2.$$

Let $x_1 = x_0 + \sqrt{2}\varepsilon b_0 v_0$ (the convenient choice of $b_0 = \pm 1$ will be specified later). Let us iterate this process: given x_k ($k \geq 1$), choose v_k so that

$$U^\varepsilon(x_k) = \max_{b=\pm 1} U^\varepsilon(x_k + \sqrt{2}\varepsilon b v_k) + \varepsilon^2$$

and let $x_{k+1} = x_k + \sqrt{2}\varepsilon b_k v_k$ where $b_k = \pm 1$ will be chosen later. For each k, we have

$$U^\varepsilon(x_k) \geq U^\varepsilon(x_{k+1}) + \varepsilon^2. \tag{3.5}$$

Combining these inequalities, we have

$$U^\varepsilon(x_0) - U^\varepsilon(x_k) \geq k\varepsilon^2. \tag{3.6}$$

We claim that if the b_k's are chosen properly then

$$U(x_{k+1}) - U(x_k) \geq -\varepsilon^2 + O(\varepsilon^3). \tag{3.7}$$

Recall that by Taylor expansion, as in Lemma 2.2, we have for any unit-norm v,

$$\max_{b_k=\pm 1} U(x_k + \sqrt{2}\varepsilon b_k v) - U(x_k) = \sqrt{2}\varepsilon |\nabla U(x_k) \cdot v| + \varepsilon^2 \langle D^2 U(x_k) v, v\rangle + O(\varepsilon^3). \tag{3.8}$$

Let us assume for the moment that $\nabla U(x_k) \neq 0$ and write temporarily $v = \cos\theta \frac{\nabla U(x_k)}{|\nabla U(x_k)|} + \sin\theta \frac{\nabla^\perp U(x_k)}{|\nabla U(x_k)|}$. Then the right hand side of (3.8) is equal to

$$\begin{aligned}
\sqrt{2}\varepsilon|\cos\theta||\nabla U(x_k)| &+ \varepsilon^2 \cos^2\theta \langle D^2 U(x_k) \frac{\nabla U(x_k)}{|\nabla U(x_k)|}, \frac{\nabla U(x_k)}{|\nabla U(x_k)|}\rangle \\
&+ \varepsilon^2 \sin^2\theta \langle D^2 U(x_k) \frac{\nabla^\perp U(x_k)}{|\nabla U(x_k)|}, \frac{\nabla^\perp U(x_k)}{|\nabla U(x_k)|}\rangle \\
&+ 2\varepsilon^2 \sin\theta\cos\theta \langle D^2 U(x_k) \frac{\nabla U(x_k)}{|\nabla U(x_k)|}, \frac{\nabla^\perp U(x_k)}{|\nabla U(x_k)|}\rangle + O(\varepsilon^3).
\end{aligned}$$

It follows using (1.5) that

$$\begin{aligned}
\max_{b_k=\pm 1} U(x_k + \sqrt{2}\varepsilon b_k v) - U(x_k) &\geq \sqrt{2}\varepsilon|\cos\theta||\nabla U(x_k)| \\
&+ \varepsilon^2 \cos^2\theta \langle D^2 U(x_k) \frac{\nabla U(x_k)}{|\nabla U(x_k)|}, \frac{\nabla U(x_k)}{|\nabla U(x_k)|}\rangle \\
&- 2\varepsilon^2 |\cos\theta| \left| \langle D^2 U(x_k) \frac{\nabla U(x_k)}{|\nabla U(x_k)|}, \frac{\nabla^\perp U(x_k)}{|\nabla U(x_k)|}\rangle \right| - \varepsilon^2 + O(\varepsilon^3).
\end{aligned} \tag{3.9}$$

We observe that the right-hand side is a polynomial of degree 2 of $|\cos\theta| \in [0,1]$, so we we see that there exists a (large) constant c_1 (depending on k) such that $|\nabla U(x_k)| \geq c_1\varepsilon$ implies that for every θ, the expression on the right-hand side is bounded below by $-\varepsilon^2 + O(\varepsilon^3)$. On the other hand, if $|\nabla U(x_k)| \leq c_1\varepsilon$ then x_k must be near the point x_* where U achieves its maximum. Since

U is C^3 and $D^2U(x_*) = -I$, we have $|\nabla U(x)| \geq (1/2)\|x - x_*\|$ in a neighborhood of x_*. Therefore $|\nabla U(x_k)| \leq c_1\varepsilon$ implies $\|x_k - x_*\| = O(\varepsilon)$, and (2.2) gives

$$\begin{aligned}
\max_{b_k=\pm 1} U(x_k + \sqrt{2}\varepsilon b_k v_k) &\geq U(x_k) + \varepsilon^2\langle D^2U(x_k)v_k, v_k\rangle + O(\varepsilon^3) \\
&= U(x_k) + \varepsilon^2\langle D^2U(x_*)v_k, v_k\rangle + O(\varepsilon^3) \\
&= U(x_k) - \varepsilon^2 + O(\varepsilon^3),
\end{aligned}$$

using Lemma 3.1(b) for the last step. We choose b_k achieving this max. Thus our claim (3.7) is valid for all x_k (even if $\nabla U(x_k) = 0$). Combining the relations (3.7) as k varies, we conclude that

$$U(x_0) - U(x_k) \leq k(\varepsilon^2 + O(\varepsilon^3)), \tag{3.10}$$

provided $x_k \in \Omega$.

The iteration stops when the sequence x_k leaves Ω. Since U^ε is bounded, this happens in at most $O(1/\varepsilon^2)$ steps. If $x_K \in \Omega$ but $x_{K+1} = x_K + \sqrt{2}\varepsilon b_K v_K$ lies outside Ω then our results give

$$U^\varepsilon(x_0) \geq (K+1)\varepsilon^2$$

and

$$U(x_0) \leq U(x_K) + K(\varepsilon^2 + O(\varepsilon^3)) \leq K\varepsilon^2 + O(\varepsilon)$$

since $K = O(1/\varepsilon^2)$ and $U(x_K) = O(\varepsilon)$ (because x_K is near $\partial\Omega$, and U vanishes at $\partial\Omega$). Subtracting these relations, we conclude that $U^\varepsilon(x_0) - U(x_0) \geq O(\varepsilon)$. This proves the proposition. □

Taken together, Propositions 3.1 and 3.2 establish Theorem 1. By the way, in proving the propositions we used the existence of a classical solution U of (1.5), but not its uniqueness. So Theorem 1 also gives, as a byproduct, an independent proof of uniqueness.

4 Motion by curvature, analyzed by the viscosity method

This section proves Theorem 2. Our attention is thus on the finite-horizon-time version of the game, where Paul's goal is to minimize $u_0(y(T))$ for a given "maturity" time T and a fixed "objective" function u_0. Theorem 2 asserts that the level sets of the limiting value function $u(x,t)$ execute motion by curvature backward in time.

If $u(x,t)$ is smooth enough, convergence can be proved by a verification argument similar to the one in Section 3. However, to avoid redundancy and achieve greater generality we present only the argument based on viscosity methods. The main point is to prove that $\lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ is a viscosity solution of (1.11).

As noted in the Introduction, Theorem 2 is quite close to the results in [CDK, Pa]. The proof in [CDK] uses a nonlinear semigroup framework, following [Ev1]; the one in [Pa] uses the fact that monotonicity, stability, and consistency imply convergence [BS]. We have chosen to give a different proof, based directly on the optimal control problem. We like this argument because it is quite elementary (modulo the use of a comparison result from the theory of viscosity solutions). It is, moreover, quite similar to the standard argument connecting the viscosity solution of a first-order Hamilton-Jacobi-Bellman equation with the value function of the associated control problem. Finally, this control-based argument will also be needed in Section 5.

One might expect the proof to proceed in two steps, first showing the existence of $u = \lim_{\varepsilon\to 0} u^\varepsilon$ then proving that the limit is a viscosity solution. Actually it does not proceed this way: the viscosity method proves *simultaneously* that the limit exists and that it is a viscosity solution. But it is perhaps of interest that the compactness of the functions u^ε can also be proved by a more classical (not viscosity-based) method. We give this argument in Appendix B.

In the time-dependent setting of this section there is no domain Ω; rather, the game is played in all $\mathbb{R}^2$. Our argument has no convexity hypotheses; the main requirement is that u_0 be continuous and constant outside of a compact set. These assumptions are used only to know that the viscosity solution of the limiting PDE is unique, as proved in [CGG, ESp] (see also [Gi1]).

4.1 The value function

We are interested in the value function $u^\varepsilon(x, k\varepsilon^2)$, defined by (1.9). For $\varepsilon > 0$ the maturity time is $T_\varepsilon = N\varepsilon^2$ with $N = N_\varepsilon$ chosen so that $\lim_{\varepsilon\to 0} T_\varepsilon = T$. Our main tool is the dynamic programming principle, already stated as (1.10), which we restate here:

$$u^\varepsilon(x, k\varepsilon^2) = \min_{\|v\|=1} \max_{b=\pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon bv, (k+1)\varepsilon^2) \tag{4.1}$$

with $u^\varepsilon(x, N\varepsilon^2) = u_0(x)$. Here the min is achieved, because u^ε is continuous in x. (The proof of Proposition B.1 in Appendix B shows that u^ε is Lipschitz continuous in x if u_0 is Lipschitz continuous; one can show by the same method that any modulus of continuity for u_0 is also a modulus of continuity for u^ε.)

4.2 Mean curvature flow

We recall that the level-set approach to mean-curvature flow (in any dimension) consists of solving (1.11) backward in time. For the following definition refer also to [CIL].

Definition 2 ([ESp]) *1. An upper semicontinuous function u is a viscosity subsolution of (1.11) if whenever $\phi(x,t)$ is smooth and $u - \phi$ has a local maximum at (x_0, t_0) we have*

$$\partial_t \phi + \Delta\phi - \langle D^2\phi \frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\rangle \geq 0 \ \textit{at}\ (x_0, t_0) \tag{4.2}$$

if $\nabla\phi(x_0, t_0) \neq 0$, and

$$\partial_t \phi + \sum_{ij} (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_i x_j} \geq 0 \ \textit{at}\ (x_0, t_0) \tag{4.3}$$

for some η with $\|\eta\| \leq 1$, if $\nabla\phi(x_0, t_0) = 0$.

2. A lower semicontinuous function u is a viscosity supersolution of (1.11) if whenever $\phi(x,t)$ is smooth and $u - \phi$ has a local minimum at (x_0, t_0) then

$$\partial_t \phi + \Delta\phi - \langle D^2\phi \frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\rangle \leq 0 \ \textit{at}\ (x_0, t_0) \tag{4.4}$$

if $\nabla\phi(x_0,t_0)\neq 0$*, and*

$$\partial_t \phi + \sum_{ij} (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_i x_j} \leq 0 \text{ at } (x_0,t_0) \tag{4.5}$$

for some η *with* $\|\eta\|\leq 1$*, if* $\nabla\phi(x_0,t_0)=0$*.*

3. u is a viscosity solution of (1.11) if it is both a sub and a supersolution.

A slightly different but equivalent definition is given in [CGG]. As we already mentioned, this viscosity solution has been proved to be unique in [ESp, CGG].

4.3 Proof of convergence

Notice that if the objective function u_0 is uniformly bounded in $\mathbb{R}^2$ then so are the value functions u^ε for all ε. Following the literature on viscosity solution, we define

$$\overline{u} \;=\; \limsup{}^* u^\varepsilon \quad \text{i.e. } \overline{u}(x) = \limsup_{y\to x,\,\varepsilon\to 0} u^\varepsilon(y) \tag{4.6}$$

$$\underline{u} \;=\; \liminf{}^* u^\varepsilon \quad \text{i.e. } \underline{u}(x) = \liminf_{y\to x,\,\varepsilon\to 0} u^\varepsilon(x). \tag{4.7}$$

Clearly $\overline{u}$ is upper semicontinuous, $\underline{u}$ is lower semicontinuous and $\underline{u}\leq\overline{u}$. We will prove that $\overline{u}$ is a viscosity subsolution (in the sense of Definition 2) and $\underline{u}$ a viscosity supersolution.

Proposition 4.1 $\overline{u}$ *is a viscosity subsolution of (1.11).*

Proof: We argue by contradiction. If not, then there is a smooth ϕ such that (x_0,t_0) is a local maximum of $\overline{u}-\phi$, with

$$\partial_t\phi + \Delta\phi - \langle D^2\phi \frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\rangle \leq \theta_0 < 0 \text{ at } (x_0,t_0) \tag{4.8}$$

if $\nabla\phi(x_0,t_0)\neq 0$, or

$$\partial_t \phi + \sum_{i,j} (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_i x_j} \leq \theta_0 < 0 \text{ at } (x_0,t_0) \tag{4.9}$$

for all η with $\|\eta\|\leq 1$, if $\nabla\phi(x_0,t_0)=0$. Adding to ϕ a nonnegative function whose derivatives at (x_0,t_0) are all zero up to second order, we can assume that (x_0,t_0) is a strict local maximum of $\overline{u}-\phi$.

Changing θ_0 if necessary, we can find a δ-neighborhood of (x_0,t_0) in which (4.8)-(4.9) hold, and in which (x_0,t_0) is the unique maximum of $\overline{u}-\phi$. We can also find, up to extraction, a sequence $(x_\varepsilon^0,t_\varepsilon^0)\to(x_0,t_0)$ such that $u^\varepsilon(x_\varepsilon^0,t_\varepsilon^0)\to\overline{u}(x_0,t_0)$. Assume for the moment that $\nabla\phi\neq 0$ in the δ-neighborhood under consideration. We construct the following sequence (which depends on ε)

$$\begin{aligned} &X_0 = (x_\varepsilon^0, t_\varepsilon^0)\\ &X_1 = \left(x_\varepsilon^0 + \sqrt{2}\varepsilon b_0 \tfrac{\nabla^\perp\phi(x_\varepsilon^0,t_\varepsilon^0)}{|\nabla\phi(x_\varepsilon^0,t_\varepsilon^0)|}, t_\varepsilon^0+\varepsilon^2\right)\end{aligned}$$

and by induction

$$X_{k+1} = X_k + \left(\sqrt{2}\varepsilon b_k \frac{\nabla^\perp \phi(X_k)}{|\nabla\phi(X_k)|}, \varepsilon^2\right).$$

Here the $b_k = \pm 1$ are to be determined later. Finally the continuous path $X(s)$ is taken to be the path that affinely interpolates between these points i.e. $X(s) = X_k + \left(\frac{s}{\varepsilon^2} - k\right)(X_{k+1} - X_k)$ for $t_\varepsilon^0 + k\varepsilon^2 \le s \le t_\varepsilon^0 + (k+1)\varepsilon^2$, and we write $X(s) = (x(s), s)$.

Using the characterization (4.1) we have, for all v such that $\|v\| = 1$,

$$u^\varepsilon(x, t_\varepsilon^0 + k\varepsilon^2) \le \max_{b=\pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon b v, t_\varepsilon^0 + (k+1)\varepsilon^2). \tag{4.10}$$

Applying this to X_k and $v = \frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}$, we can now choose the b_k's that achieve the max in all the quantities above and get

$$u^\varepsilon(X_k) \le u^\varepsilon(X_{k+1}). \tag{4.11}$$

Summing up these inequalities, we are led to

$$u^\varepsilon(X_0) \le u^\varepsilon(X_k). \tag{4.12}$$

Next, let us evaluate $\phi(X(t_0 + s)) - \phi(X(t_0))$. We have

$$\frac{\partial}{\partial t}\phi(X(t)) = \partial_t \phi(x(t), t) + \nabla\phi(x(t), t) \cdot d_t x(t). \tag{4.13}$$

But

$$\frac{\partial}{\partial t}\nabla\phi(x(t), t) = \nabla\phi_t(x(t), t) + D^2\phi(x(t), t) d_t x(t)$$

hence when $d_t x(t)$ is a constant, we have

$$\frac{\partial^2}{\partial t^2}\phi(X(t)) = \partial_t^2\phi(X(t)) + 2\nabla\phi_t(X(t)) \cdot d_t x(t) + \langle D^2\phi(X(t)) d_t x(t), d_t x(t)\rangle. \tag{4.14}$$

Observing that for $s \in [t_\varepsilon^0 + k\varepsilon^2, t_\varepsilon^0 + (k+1)\varepsilon^2)$, we have $d_t x(t) = b_k \frac{\sqrt{2}}{\varepsilon} \frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}$, we deduce, writing a Taylor expansion of $\phi(X(t))$ around $t_\varepsilon^0 + k\varepsilon^2$, that

$$\begin{aligned}\phi(X_{k+1}) - \phi(X_k) = \varepsilon^2 \Big(\partial_t\phi(X_k) + \nabla\phi(X_k) \cdot b_k \frac{\sqrt{2}}{\varepsilon}\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}\Big) \\ + \frac{\varepsilon^4}{2}\Big(\partial_t^2\phi(X_k) + 2\nabla\phi_t(X_k) \cdot b_k \frac{\sqrt{2}}{\varepsilon}\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|} \\ + \frac{2}{\varepsilon^2}\langle D^2\phi(X_k)\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}, \frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}\rangle\Big) + o(\varepsilon^2).\end{aligned} \tag{4.15}$$

Using the fact that ϕ is smooth, we deduce that

$$\phi(X_{k+1}) - \phi(X_k) = \varepsilon^2 \left(\partial_t\phi(X_k) + \langle D^2\phi(X_k)\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}, \frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}\rangle + o(1)\right), \tag{4.16}$$

where the $o(1)$ depends only on the C^3 norm of ϕ. Then, with the hypothesis (4.8), and using the identity

$$\Delta\phi - \langle D^2\phi\frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\rangle = \langle D^2\phi\frac{\nabla^\perp\phi}{|\nabla\phi|}, \frac{\nabla^\perp\phi}{|\nabla\phi|}\rangle,$$

we deduce that

$$\phi(X_{k+1}) - \phi(X_k) \leq \varepsilon^2\theta_0 + o(\varepsilon^2). \tag{4.17}$$

Adding up all these estimates, we finally obtain that

$$\phi(X_k) - \phi(X_0) \leq k\varepsilon^2(\theta_0 + o(1)) < \frac{k\varepsilon^2\theta_0}{2} < 0 \quad \text{for } \varepsilon \text{ small enough.} \tag{4.18}$$

Adding this relation to (4.12), we obtain

$$u^\varepsilon(X_0) - \phi(X_0) \leq u^\varepsilon(X_k) - \phi(X_k). \tag{4.19}$$

Now, recall that $X_0 = (x_\varepsilon^0, t_\varepsilon^0) \to (x_0, t_0)$ and $u^\varepsilon(X_0) \to \overline{u}(x_0, t_0)$ by construction, thus $\phi(X_0) \to \phi(x_0, t_0)$ (by continuity of ϕ). It is tempting to take k of order $1/\varepsilon^2$, so that X_k stays bounded away from X_0, by (4.18). But we cannot necessarily do this: since the spatial step size is ε, X_k can easily leave our δ-neighborhood of X_0 for k of order $1/\varepsilon$. No matter: we can certainly choose $k = k_\varepsilon$ so that X_k stays in the δ-neighborhood of (x_0, t_0), and $X_k \to (x', t') \neq (x_0, t_0)$ (after extraction). We then have $\lim \phi(X_k) = \phi(x', t')$ and obviously by definition of $\overline{u}$, $\limsup_{\varepsilon\to 0} u^\varepsilon(X_k) \leq \overline{u}(x', t')$. Combining all these elements and passing to the limit in (4.19) we are led to

$$\overline{u}(x_0, t_0) - \phi(x_0, t_0) \leq \overline{u}(x', t') - \phi(x', t'). \tag{4.20}$$

Since (x', t') is not equal to (x_0, t_0) and is in the δ-neighborhood in which (x_0, t_0) is the unique maximum of $\overline{u} - \phi$, we have reached a contradiction.

We assumed above that $\nabla\phi(x_0, t_0) \neq 0$. But a similar argument can be used if $\nabla\phi(x_0, t_0) = 0$. Indeed: we use the same construction whenever $\nabla\phi(X_k) \neq 0$, and we replace $\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}$ by an arbitrary unit-norm vector whenever $\nabla\phi(X_k) = 0$. Observe that when $\|\eta\|^2 = 1$,

$$\sum_{ij} \phi_{x_i x_j}(\delta_{ij} - \eta_i\eta_j) = \langle D^2\phi\eta^\perp, \eta^\perp\rangle;$$

therefore (4.9) says that for every v such that $\|v\| = 1$,

$$\partial_t\phi + \langle D^2\phi v, v\rangle \leq \theta_0 < 0 \tag{4.21}$$

in the δ-neighborhood of (x_0, t_0). Using this fact and arguing exactly as above, we once again reach a contradiction. $\square$

Proposition 4.2 *$\underline{u}$ is a viscosity supersolution of (1.11).*

Proof: We argue by contradiction. If not, then there is a smooth ϕ such that (x_0, t_0) is a local minimum of $\underline{u} - \phi$, with

$$\partial_t\phi + \Delta\phi - \langle D^2\phi\,\frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\rangle \geq \theta_0 > 0 \text{ at } (x_0, t_0) \tag{4.22}$$

if $\nabla\phi(x_0, t_0) \neq 0$ and

$$(4.23) \qquad \partial_t\phi + \sum_{ij} (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_ix_j} \geq \theta_0 > 0 \text{ at } (x_0, t_0)$$

for all η with $\|\eta\| \leq 1$, if $\nabla\phi(x_0, t_0) = 0$. Again, without loss of generality, we can assume the minimum is strict. Changing θ_0 if necessary, we can also find a δ-neighborhood of (x_0, t_0) in which such assertions hold. Again we can find $(x_\varepsilon^0, t_\varepsilon^0) \to (x_0, t_0)$ such that $u^\varepsilon(x_\varepsilon^0, t_\varepsilon^0) \to \underline{u}(x_0, t_0)$. Taking the unit-norm v_0 that achieves the minimum in the characterization (4.1), we find

$$(4.24) \qquad u^\varepsilon(x_\varepsilon^0, t_\varepsilon^0) = \max_{b=\pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon b v_0, t_\varepsilon^0 + \varepsilon^2).$$

Let thus $X_0 = (x_\varepsilon^0, t_\varepsilon^0)$, $X_1 = (x_\varepsilon^0 + \sqrt{2}\varepsilon b_0 v_0, t_\varepsilon^0 + \varepsilon^2)$ and inductively $X_{k+1} = X_k + (\sqrt{2}\varepsilon b_k v_k, \varepsilon^2)$ where the v_k achieve the min in (4.1) and the b_k are to be determined later. We have for all $b_k = \pm 1$,

$$u^\varepsilon(X_k) \geq u^\varepsilon(X_{k+1}),$$

and thus

$$(4.25) \qquad u^\varepsilon(X_0) \geq u^\varepsilon(X_k).$$

On the other hand, extending X_k into an affine path by affine interpolation, and doing a Taylor expansion as in (4.15), we find

$$(4.26) \qquad \phi(X_{k+1}) - \phi(X_k) = \varepsilon\nabla\phi(X_k)\cdot b_k v_k + \varepsilon^2\left(\partial_t\phi(X_k) + \langle D^2\phi(X_k)v_k, v_k\rangle\right) + O(\varepsilon^3).$$

Let us first consider the case where $\nabla\phi(x_0, t_0) \neq 0$. We can assume that $\nabla\phi(x, t) \neq 0$ for all (x, t) in the δ-neighborhood of (x_0, t_0) we are working in. Then, we use the analogue of Lemma 2.2(c) (the proof is exactly the same) to see that for all sufficiently small ε the following statement is true: for every $v \in \mathbb{R}^2$ such that $\|v\| = 1$, there exists a $b = \pm 1$ such that

$$(4.27) \qquad \phi(x + \sqrt{2}\varepsilon b v, t + \varepsilon^2) - \phi(x, t) \\ \geq \varepsilon^2\left(\partial_t\phi(x, t) + \langle D^2\phi(x, t)\frac{\nabla^\perp\phi(x, t)}{|\nabla\phi(x, t)|}, \frac{\nabla^\perp\phi(x, t)}{|\nabla\phi(x, t)|}\rangle\right) - C\left(1 + \frac{1}{|\nabla\phi(x, t)|}\right)\varepsilon^3.$$

Using the assumption (4.22) (which is assumed to hold in the whole δ-neighborhood of (x_0, t_0)), we are led to

$$(4.28) \qquad \phi(X_{k+1}) - \phi(X_k) \geq \varepsilon^2\frac{\theta_0}{2},$$

for ε small enough.

When $\nabla\phi(x_0, t_0) = 0$, we simply write that by the analogue of Lemma 2.2(a), for any $\|v\| = 1$ there exists $b = \pm 1$ such that

$$(4.29) \qquad \phi(x + \sqrt{2}\varepsilon b v, t + \varepsilon^2) - \phi(x, t) \geq \varepsilon^2\left(\partial_t\phi(x, t) + \langle D^2\phi(x, t)v, v\rangle\right) + O(\varepsilon^3).$$

and using (4.23), we deduce that (4.28) holds as well. Combining these relations as k varies, we find

$$(4.30) \qquad \phi(X_k) - \phi(X_0) \geq k\varepsilon^2\frac{\theta_0}{2},$$

and adding this relation to (4.25), we are led to

$$u^\varepsilon(X_0) - \phi(X_0) \geq u^\varepsilon(X_k) - \phi(X_k). \tag{4.31}$$

Arguing as in the proof of the previous proposition, we have $u^\varepsilon(X_0) - \phi(X_0) \to \underline{u}(x_0, t_0) - \phi(x_0, t_0)$ as $\varepsilon \to 0$. We can choose $k = k_\varepsilon$ so that $X_k \to (x', t')$ belongs to the δ-neighborhood of (x_0, t_0) but is different from (x_0, t_0). Passing to the limit in (4.31) and using the fact that $\liminf u^\varepsilon(X_k) \geq \underline{u}(x', t')$, we find

$$\underline{u}(x_0, t_0) - \phi(x_0, t_0) \geq \underline{u}(x', t') - \phi(x', t') \tag{4.32}$$

which contradicts the assumption of strict local minimality of (x_0, t_0). $\square$

Combining these two propositions, we obtain that $\overline{u}$ is a subsolution and $\underline{u}$ a supersolution, thus from maximum (or uniqueness) principles for viscosity solutions, we must have $\overline{u} \leq \underline{u}$. On the other hand, by construction, $\underline{u} \leq \liminf^* u_\varepsilon \leq \limsup^* u^\varepsilon \leq \overline{u}$, thus $\underline{u} = \overline{u}$, it can be denoted u and is the unique viscosity solution of (1.11). This also implies by elementary arguments (see [CIL] or [Barles] Lemma 4.1 p 86) that u^ε converges uniformly to this u on every compact subset of $\mathbb{R}^2 \times \mathbb{R}$. Thus we have proved Theorem 2.

5 The minimum exit time, for nonconvex domains in the plane

This section considers nonconvex domains, developing a link between the minimum exit time and "objective function" approaches. As explained in the introduction, this link involves relaxing the constraint $\|v\| = 1$ to $\|v\| \leq 1$ in the game. It is now necessary to assume that the objective function u_0 is positive in Ω, 0 on $\partial\Omega$, and nonpositive in $\mathbb{R}^2 \backslash \Omega$.

5.1 Convergence of the time-horizon game

The time horizon game with $\|v\| \leq 1$ is defined as in (1.10) except that Paul is allowed to choose a vector $\|v\| \leq 1$. The dynamic programming characterization then becomes $u^\varepsilon(x, T) = u_0(x)$ and

$$u^\varepsilon(x, t) = \min_{\|v\| \leq 1} \max_{b = \pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon b v, t + \varepsilon^2) \tag{5.1}$$

The solution of the positive mean curvature flow is defined by the analogue of (1.11) (see [CGG]) with $|\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ replaced by $\left(|\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right)_-$ where f_- denotes the negative part of f. This equation also has a unique solution in the viscosity sense. It corresponds to the evolution of the level sets with normal velocity equal to the positive part of their curvature. A restatement of Theorem 5 is this:

Theorem 5 *As $\varepsilon \to 0$, the value function u^ε for the $\|v\| \leq 1$ game with objective-function u_0 and maturity time T converges locally uniformly to u, the viscosity solution of*

$$\begin{cases} u_t + \left(|\nabla u| \mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)\right)_- = 0 \\ u(x, T) = u_0(x) \end{cases} \tag{5.2}$$

The proof is like that of Theorem 2, with a few minor adjustments. First: in proving the analogue of Proposition 4.1, when $\nabla\phi(x_0, t_0) \neq 0$ and $\langle D^2\phi(X_k)\frac{\nabla^\perp\phi(X_k)}{|\nabla^\perp\phi(X_k)|}, \frac{\nabla^\perp\phi(X_k)}{|\nabla^\perp\phi(X_k)|}\rangle > 0$ we must replace the previous choice of X_{k+1} by $X_k + (0, \varepsilon^2)$. Second, the relevant analogue of Lemma 2.2(c) is the following: for every $v \in \mathbb{R}^2$ such that $\|v\| \leq 1$, there exists a $b = \pm 1$ such that

$$\phi(x + \sqrt{2}\varepsilon bv) - \phi(x) \geq \varepsilon^2 \left(\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle \right)_- - C\left(1 + \frac{1}{|\nabla\phi(x)|}\right)\varepsilon^3. \tag{5.3}$$

This follows from the fact that

$$\min_{\|v\|\leq 1, v\cdot\nabla\phi=0} \langle D^2\phi(x)v, v\rangle = \left(\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla\phi(x)|}\rangle \right)_- . \tag{5.4}$$

Third, if $\nabla\phi(x_0, t_0) = 0$, then by Taylor expansion we have

$$\phi(x + \sqrt{2}\varepsilon bv) - \phi(x) \geq \varepsilon^2 \left(\langle D^2\phi(x)v, v\rangle\right)_- + O(\varepsilon^3) \tag{5.5}$$

for any $\|v\| \leq 1$. Finally, (5.3)-(5.5) have obvious time-dependent analogues. Using these observations, the proof of Theorem 5 is straightforward.

5.2 Equivalence of the time-horizon and exit games

Now consider the exit-time version of the $\|v\| \leq 1$ game. The definition of the exit time or value function U^ε is like Definition 1, except that the "unit-norm vector v" is now a vector of norm ≤ 1. The dynamic programming characterization of U^ε is

$$U^\varepsilon(x) = \min_{\|v\|\leq 1} \max_{b=\pm 1} U^\varepsilon(x + \sqrt{2}\varepsilon bv) + \varepsilon^2. \tag{5.6}$$

The following result asserts, in essence, that the exit-time and time-horizon versions of the $\|v\| \leq 1$ game are equivalent.

Lemma 5.1 *Let u^ε and U^ε be the value functions of the time horizon game and exit-time games with $\|v\| \leq 1$. Then, for all $x \in \overline{\Omega}$, and all $t = k\varepsilon^2$,*

$$U^\varepsilon(x) \leq t \Longleftrightarrow u^\varepsilon(x, T - t) \leq 0. \tag{5.7}$$

Proof: Observe first that since we can take $v = 0$ as a test vector in (5.1), we have for every x, t,

$$u^\varepsilon(x, t) = \min_{\|v\|\leq 1} \max_{b=\pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon bv, t + \varepsilon^2) \leq u^\varepsilon(x, t + \varepsilon^2), \tag{5.8}$$

that is for fixed x, u^ε increases in time. (This is different from the game with $\|v\| = 1$.)

Let us prove by induction on k that if $x \in \overline{\Omega}$, $U^\varepsilon(x) \leq k\varepsilon^2$ implies $u^\varepsilon(x, T - k\varepsilon^2) \leq 0$. The assertion is true for $k = 0$ because if $U^\varepsilon(x) = 0$ and $x \in \overline{\Omega}$ then $x \in \partial\Omega$ and thus by definition $u^\varepsilon(x, T) = u_0(x) = 0$. Assume it is true for k, and $U^\varepsilon(x) \leq (k+1)\varepsilon^2$. Then, by characterization (5.6),

$$\min_{\|v\|\leq 1} \max_{b=\pm 1} U^\varepsilon(x + \sqrt{2}\varepsilon bv) \leq k\varepsilon^2.$$

That is there exists v with $\|v\| \leq 1$, such that for all $b = \pm 1$, $U^\varepsilon(x+\sqrt{2}\varepsilon bv) \leq k\varepsilon^2$. If $x+\sqrt{2}\varepsilon bv \notin \Omega$, then by definition

$$u^\varepsilon(x+\sqrt{2}\varepsilon bv, T) = u_0(x+\sqrt{2}\varepsilon bv) \leq 0$$

and thus for all k, $u^\varepsilon(x+\sqrt{2}\varepsilon bv, T-k\varepsilon^2) \leq 0$ in view of (5.8). If on the other hand $x+\sqrt{2}\varepsilon bv \in \Omega$, then by the induction hypothesis we can also deduce that $u^\varepsilon(x+\sqrt{2}\varepsilon bv, T-k\varepsilon^2) \leq 0$ holds for this v and every b. Thus,

$$\min_{\|v\|\leq 1} \max_{b=\pm 1} u^\varepsilon(x+\sqrt{2}\varepsilon bv, T-k\varepsilon^2) \leq 0$$

which is exactly

$$u^\varepsilon(x, T-(k+1)\varepsilon^2) \leq 0.$$

The property is thus proved by induction.

Conversely, let us prove by induction that if $x \in \overline{\Omega}$, $u^\varepsilon(x, T-k\varepsilon^2) \leq 0$ implies $U^\varepsilon(x) \leq k\varepsilon^2$. The property is clearly true for $k=0$. Assume it is true for k, and $u^\varepsilon(x, T-(k+1)\varepsilon^2) \leq 0$. Then by (5.1), there exists $\|v\| \leq 1$ such that for every $b = \pm 1$, $u^\varepsilon(x+\sqrt{2}\varepsilon bv, T-k\varepsilon^2) \leq 0$. By the induction hypothesis, for all $b = \pm 1$, $U^\varepsilon(x+\sqrt{2}\varepsilon bv) \leq k\varepsilon^2$ (this is trivially valid when $x+\sqrt{2}\varepsilon bv \notin \Omega$). We deduce that $\min_{\|v\|\leq 1} \max_{b=\pm 1} U^\varepsilon(x+\sqrt{2}\varepsilon bv) \leq k\varepsilon^2$, thus by (5.6), $U^\varepsilon(x) \leq (k+1)\varepsilon^2$. The property is proved by induction. □

Remark : The exit-time and time-horizon versions of the $\|v\| = 1$ game are *not* equivalent.

Theorem 6 is an easy consequence of Lemma 5.1.

Proof of Theorem 6: First, we define as before

$$\begin{aligned} \overline{U} &= \limsup{}^* U^\varepsilon \quad \text{i.e. } \overline{U}(x) = \limsup_{y\to x,\, \varepsilon\to 0} U^\varepsilon(y) \\ \underline{U} &= \liminf{}^* U^\varepsilon \quad \text{i.e. } \underline{U}(x) = \liminf_{y\to x,\, \varepsilon\to 0} U^\varepsilon(x). \end{aligned}$$

From Lemma 5.1, we get that $u^\varepsilon(x, T-U^\varepsilon(x)) \leq 0$. Since u^ε converges uniformly on compact subsets of Ω, we deduce that for every $x \in \Omega$, $u(x, T-\overline{U}(x)) \leq 0$ and $u(x, T-\underline{U}(x)) \leq 0$. On the other hand from Lemma 5.1 we also have $U^\varepsilon(x) > t \Leftrightarrow u^\varepsilon(x, T-t) > 0$. Therefore taking $t = U^\varepsilon - \varepsilon^2$ we have $u^\varepsilon(x, T-U^\varepsilon(x)+\varepsilon^2) > 0$, and passing to the limit $\varepsilon \to 0$ gives $u(x, T-\overline{U}(x)) \geq 0$ and $u(x, T-\underline{U}(x)) \geq 0$ by uniform convergence of u^ε. Thus

$$u(x, T-\underline{U}(x)) = u(x, T-\overline{U}(x)) = 0,$$

as asserted. □

5.3 Convergence of the exit-time games

We have proved Theorems 5 and 6. However we have yet to address Theorems 3 and 8, concerning the convergence of the exit times U^ε for the $\|v\| = 1$ and $\|v\| \leq 1$ games. Recall that by Lemma 2.1 these exit times are uniformly bounded in $\overline{\Omega}$. The interesting case is of course when Ω is not convex.

We use the framework of viscosity solutions of boundary-value problems. Since the Hamiltonians under consideration are not smooth enough, the usual definition (see [CIL, Barles]) needs to be relaxed as in [ESp] or via "superjets" (see [CIL]). Here is a careful definition of what it means for u to be a subsolution or supersolution of (1.5).

Definition 3 *1. An upper semicontinuous function U is a viscosity subsolution of*

$$\left\{ \begin{array}{rcl} -1 - |\nabla U| \mathrm{div}\left(\frac{\nabla U}{|\nabla U|}\right) & = & 0 \quad \text{in } \Omega \\ U & = & 0 \quad \text{at } \partial\Omega \end{array} \right. \tag{5.9}$$

if

(a) *whenever ϕ is smooth and $U - \phi$ has a local maximum at $x_0 \in \Omega$ then*

$$1 + \Delta\phi - \langle D^2\phi \frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\rangle \geq 0 \text{ at } x_0$$

if $\nabla\phi(x_0) \neq 0$, and

$$1 + \sum_{ij} (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_i x_j} \geq 0 \text{ at } x_0$$

for some η with $\|\eta\| \leq 1$, if $\nabla\phi(x_0) = 0$; and

(b) *whenever ϕ is smooth and $U-\phi$ has a local maximum at $x_0 \in \partial\Omega$, either the previous condition holds or $U(x_0) \leq 0$.*

2. *A lower semicontinuous function U is a viscosity supersolution of (5.9) if*

(a) *whenever ϕ is smooth and $U - \phi$ has a local minimum at $x_0 \in \Omega$ then*

$$1 + \Delta\phi - \langle D^2\phi \frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\rangle \leq 0 \text{ at } x_0$$

if $\nabla\phi(x_0) \neq 0$, and

$$1 + \sum_{ij} (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_i x_j} \leq 0 \text{ at } x_0$$

for some η with $\|\eta\| \leq 1$, if $\nabla\phi(x_0) = 0$; and

(b) *whenever ϕ is smooth and $U-\phi$ has a local minimum at $x_0 \in \partial\Omega$, either the previous condition holds or else $U(x_0) \geq 0$.*

Our goal is to prove Theorems 3 and 8. Here is a rigorous statement of their combined assertions.

Theorems 3 and 8 *For either the $\|v\| = 1$ or $\|v\| \leq 1$ game, let U^ε be the scaled exit time, and consider*

$$\begin{array}{rcl} \overline{U} & = & \limsup{}^* U^\varepsilon \quad \text{i.e. } \overline{U}(x) = \limsup\limits_{y\to x,\, \varepsilon\to 0} U^\varepsilon(y) \\ \underline{U} & = & \liminf{}^* U^\varepsilon \quad \text{i.e. } \underline{U}(x) = \liminf\limits_{y\to x,\, \varepsilon\to 0} U^\varepsilon(x). \end{array}$$

Then $\overline{U}$ is a viscosity subsolution of (5.9) and $\underline{U}$ is a viscosity supersolution of (5.9), as defined in Definition 3.

Proof: The overall strategy is the same as for Theorem 2. We focus first, in Steps 1 and 2, on the $\|v\|=1$ version of the game. Then in Step 3 we'll consider the $\|v\|\leq 1$ game.

STEP 1. We claim that when U^ε is the exit time for the the $\|v\|=1$ game, $\overline{U}$ is a viscosity subsolution of (5.9). Clearly it is upper semicontinuous. We argue by contradiction. Suppose ϕ is smooth and x_0 is a local maximum of $\overline{U}-\phi$ (with $x_0\in\Omega$ for the moment) such that

$$1+|\nabla\phi|\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)=1+\langle D^2\phi\,\frac{\nabla^\perp\phi}{|\nabla\phi|},\frac{\nabla^\perp\phi}{|\nabla\phi|}\rangle\leq\theta_0<0 \text{ at } x_0, \tag{5.10}$$

if $\nabla\phi(x_0)\neq 0$, and

$$1+\sum_{ij}\left(\delta_{ij}-\eta_i\eta_j\right)\phi_{x_ix_j}\leq\theta_0<0 \tag{5.11}$$

for all η with $\|\eta\|\leq 1$, if $\nabla\phi(x_0,t_0)=0$. Adding to ϕ a nonnegative function whose derivatives at x_0 vanish to second order, we can assume that x_0 is a strict local maximum of $\overline{U}-\phi$.

Changing θ_0 if necessary, we can find a δ-neighborhood of x_0 in Ω in which (5.10)-(5.11) hold, and in which x_0 is the unique maximum of $\overline{U}-\phi$. We can also find, up to extraction, a sequence $x_\varepsilon^0\to x_0$ such that $U^\varepsilon(x_\varepsilon^0)\to\overline{U}(x_0)$. Assume for the moment that $\nabla\phi\neq 0$ in the δ-neighborhood under consideration. We construct the following sequence

$$\begin{aligned}&X_0=x_\varepsilon^0\\&X_{k+1}=X_k+\sqrt{2}\varepsilon b_k\tfrac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}.\end{aligned}$$

Here the $b_k=\pm1$ are to be determined later. Finally the continuous path $X(s)$ is taken to be the path that affinely interpolates between those points, i.e. $X(s)=X_k+\left(\frac{s}{\varepsilon^2}-k\right)(X_{k+1}-X_k)$ for $k\varepsilon^2\leq s\leq(k+1)\varepsilon^2$.

Using the characterization (2.1), we can choose the b_k's such that

$$U^\varepsilon(X_k)\leq U^\varepsilon(X_{k+1})+\varepsilon^2. \tag{5.12}$$

Summing up these inequalities over k, we are led to

$$U^\varepsilon(X_0)\leq U^\varepsilon(X_k)+k\varepsilon^2. \tag{5.13}$$

On the other hand, by Taylor expansion, we have

$$\phi(X_{k+1})-\phi(X_k)=\varepsilon^2<D^2\phi(X_k)\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|},\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}>+o(\varepsilon^2) \tag{5.14}$$

and from (5.10), we deduce that

$$\phi(X_{k+1})-\phi(X_k)\leq\varepsilon^2(-1+\theta_0)+o(\varepsilon^2). \tag{5.15}$$

Adding up all these estimates, we obtain that

$$\phi(X_k)-\phi(X_0)\leq k\varepsilon^2(-1+\theta_0+o(1))<-k\varepsilon^2\quad\text{for }\varepsilon\text{ small enough.} \tag{5.16}$$

Now combining (5.13) and (5.16) we conclude that

$$U^\varepsilon(X_0) - \phi(X_0) \le U^\varepsilon(X_k) - \phi(X_k). \tag{5.17}$$

Recall that $X_0 = x_\varepsilon^0 \to x_0$ and $u^\varepsilon(X_0) \to \overline{U}(x_0, t_0)$ by construction, and $\phi(X_0) \to \phi(x_0)$ by the continuity of ϕ.

As in the proof of Proposition 4.1, it is tempting to take k of order $1/\varepsilon^2$, so that X_k stays bounded away from X_0 by (5.16). But we cannot necessarily do this: since the spatial step size is ε, X_k can easily leave our δ-neighborhood of X_0 for k of order $1/\varepsilon$. No matter: we can choose $k = k_\varepsilon$ so that X_k stays in the δ-neighborhood of x_0, and $X_k \to x' \ne x_0$ (after extraction). We then have $\lim \phi(X_k) = \phi(x')$ and obviously by definition of $\overline{U}$, $\limsup_{\varepsilon\to 0} U^\varepsilon(X_k) \le \overline{U}(x')$. Combining all these elements and passing to the limit in (5.17) we conclude that

$$\overline{U}(x_0) - \phi(x_0) \le \overline{U}(x') - \phi(x').$$

But x' is in the δ-neighborhood where x_0 is the unique maximum of $\overline{U} - \phi$, and $x' \ne x_0$. This is a contradiction.

We assumed above that $\nabla\phi(x_0) \ne 0$ in the δ-neighborhood of x_0. But this assumption was not really necessary. When $\nabla\phi(X_k) = 0$ we need merely replace $\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}$ by an arbitrary unit-norm vector. The argument still works, with obvious modifications, leading as before to a contradiction.

Finally, consider $x_0 \in \partial\Omega$. Arguing by contradiction as usual, we suppose (5.10)-(5.11) hold and also

$$\overline{U}(x_0) \ge \eta > 0.$$

Taking $x_\varepsilon^0 \to x_0$ such that $U^\varepsilon(x_\varepsilon^0) \to \overline{U}(x_0)$, we have $U^\varepsilon(x_\varepsilon^0) \ge \frac{\eta}{2}$ for ε small enough. This implies that $x_\varepsilon^0 \in \Omega$ (since by definition $U^\varepsilon = 0$ on $\partial\Omega$). We can apply the exact same construction as above. For $k < \frac{\eta}{2\varepsilon^2}$ we have, in view of (5.13), $U^\varepsilon(X_k) > 0$ and therefore $X_k \in \Omega$. So we can complete the argument as before (keeping $k < \frac{\eta}{2\varepsilon^2}$), obtaining a contradiction in this case too.

STEP 2: Now let's show that when U^ε is the exit time for the the $\|v\| = 1$ game, $\underline{U}$ is a viscosity supersolution of (5.9). Clearly it is lower semicontinuous. We argue by contradiction. Suppose ϕ is smooth and x_0 is a strict local minimum of $\underline{U} - \phi$ (again we can assume strictness without loss of generality) and

$$1 + \langle D^2\phi \frac{\nabla^\perp\phi}{|\nabla\phi|}, \frac{\nabla^\perp\phi}{|\nabla\phi|}\rangle \ge \theta_0 > 0 \text{ in a neighborhood of } x_0 \tag{5.18}$$

if $\nabla\phi(x_0) \ne 0$, or

$$1 + (\delta_{ij} - \eta_i\eta_j)\,\phi_{x_ix_j} \ge \theta_0 > 0 \text{ in a neighborhood of } x_0 \tag{5.19}$$

for all η with $\|\eta\| \le 1$, if $\nabla\phi(x_0) = 0$. We need only consider $x_0 \in \Omega$ since $\underline{U} = \liminf U^\varepsilon \ge 0$, so that part (b) of the definition of a supersolution is trivially satisfied.

Let $x_\varepsilon^0 \to x_0$ such that $U^\varepsilon(x_\varepsilon^0) \to \underline{U}(x_0)$. Let us construct again a sequence X_k with $X_0 = x_\varepsilon^0$ by iteration. Using the characterization (2.1), X_k being constructed, we can choose v_k and $X_{k+1} = X_k + \sqrt{2}\varepsilon b_k v_k$ such that for any choice of $b_k = \pm 1$,

$$U^\varepsilon(X_k) \ge U^\varepsilon(X_{k+1}) + \varepsilon^2. \tag{5.20}$$

Adding up those relations leads to

$$U^\varepsilon(X_0) \geq U^\varepsilon(X_k) + k\varepsilon^2. \tag{5.21}$$

On the other hand, using Lemma 2.2(c) with (5.18) if $\nabla\phi(x_0) \neq 0$, and (5.19) if $\nabla\phi(x_0) = 0$, we can choose the b_k's above in such a way that

$$\phi(X_{k+1}) \geq \phi(X_k) + \varepsilon^2(-1+\theta_0) + o(\varepsilon^2) \tag{5.22}$$

and thus

$$\phi(X_k) \geq \phi(X_0) - k\varepsilon^2. \tag{5.23}$$

Adding up (5.21) and (5.23) yields

$$U^\varepsilon(X_0) - \phi(X_0) \geq U^\varepsilon(X_k) - \phi(X_k). \tag{5.24}$$

Now take $k \leq O(\frac{1}{\varepsilon^2})$ so that X_k stays in the δ-neighborhood and (up to extraction) $X_k \to x' \neq x_0$. We have $\liminf_{\varepsilon\to 0} U^\varepsilon(X_k) \geq \underline{U}(x')$, and passing to the limit in (5.24) we get

$$\underline{U}(x_0) - \phi(x_0) \geq \underline{U}(x') - \phi(x'). \tag{5.25}$$

This is a contradiction, since x_0 was the unique minimum of $\underline{U} - \phi$ in the δ-neighborhood.

STEP 3: We turn now to the $\|v\| \leq 1$ version of the game. Arguing exactly as in Steps 1 and 2, we see that $\overline{U}$ is a subsolution and $\underline{U}$ a supersolution of

$$\begin{cases} -1 - \left(|\nabla U|\mathrm{div}\left(\frac{\nabla U}{|\nabla U|}\right)\right)_- &= 0 \quad \text{in } \Omega \\ U &= 0 \quad \text{at } \partial\Omega \end{cases} \tag{5.26}$$

No new idea is need; the main departure from the previous argument is that we must use (5.3) in place of Lemma 2.2.

The assertion of the theorem is that $\overline{U}$ and $\underline{U}$ are sub and supersolutions of (5.9) not (5.26). But actually the two equations have the *same* sub and supersolutions. This is a consequence of the elementary facts that for any $y \in \mathbb{R}$,

$$1 + y_- \geq 0 \iff y \geq -1 \iff 1 + y \geq 0 \tag{5.27}$$

and

$$1 + y_- \leq 0 \iff y \leq -1 \iff 1 + y \leq 0. \tag{5.28}$$

For example, suppose U is a subsolution of (5.26). Then for every smooth ϕ, if $U - \phi$ has a local maximum at $x_0 \in \Omega$ we have

$$1 + \left(\langle D^2\phi \frac{\nabla^\perp\phi}{|\nabla\phi|}, \frac{\nabla^\perp\phi}{|\nabla\phi|}\rangle\right)_- \geq 0 \text{ at } x_0 \tag{5.29}$$

if $\nabla\phi(x_0) \neq 0$, and

$$1 + \left(\sum_{ij} (\delta_{ij} - \eta_i \eta_j) \phi_{x_i x_j} \right)_- \geq 0 \tag{5.30}$$

for some η with $\|\eta\| \leq 1$, if $\nabla\phi(x_0) = 0$. Moreover if $U - \phi$ has a local maximum at $x_0 \in \partial\Omega$, either the preceding condition holds or $U(x_0) \leq 0$. By (5.27), (5.29) and (5.30) are equivalent to

$$1 + \langle D^2\phi \frac{\nabla^\perp \phi}{|\nabla\phi|}, \frac{\nabla^\perp \phi}{|\nabla\phi|} \rangle \geq 0 \text{ at } x_0$$

and

$$1 + \sum_{ij} (\delta_{ij} - \eta_i \eta_j)\, \phi_{x_i x_j} \geq 0$$

respectively. Therefore U is also a subsolution of (5.9). The proof that every supersolution of (5.26) is also a supersolution of (5.9) is similar, using (5.28). □

As noted in the Introduction, Theorem 4 assures us that $\overline{U}_* = \underline{U}$ and $\overline{U} = \underline{U}^*$ for star-shaped domains. This characterizes the limiting value function as the unique-up-to-envelope (possible discontinuous) viscosity solution of (5.9).

5.4 Behavior of the exit time at the boundary

This section proves Theorem 7. Recall that at finite ε, Paul can only exit from the convex part of the boundary (see Figure 3). If he starts a distance of order ε^2 from the convex part of the boundary, he should be able to exit in just a few steps; but if he starts near the concave part of the boundary he should need many steps, since he must exit at a distant location. Theorem 7 refines these two assertions, by proving analogous statements about the *scaled* exit times.

Proof of (1.19): We must show that if Paul starts near the strictly convex part of $\partial\Omega$ then his scaled exit time is small. The proof resembles the argument used for Lemma 2.1. The plan is simple: we shall specify an exit strategy for Paul. The strategy is not optimal, but this doesn't matter: it gives the desired upper bound on his minimum exit time.

Fix $x_* \in \partial\Omega$ and assume $\partial\Omega$ is strictly convex and C^2 nearby. Let R_* be the radius of curvature at x_*, so $\partial\Omega$ is well-approximated near x_* by its osculating circle, which has radius R_*. Now consider a larger circle that's also tangent to $\partial\Omega$ at x_*; let p be it's center, and $R > R_*$ its radius. If δ is sufficiently small then the concentric circle

$$\|x - p\|^2 = R^2 - \delta$$

meets $\partial\Omega$ in two points near x_* and determines a crescent-shaped neighborhood of x_* in Ω (see Figure 5; the crescent-shaped region lies between the circular arc ZW and $\partial\Omega$). We claim that *if Paul starts in this crescent-shaped region then he requires at most $\frac{1}{2}\delta\varepsilon^{-2}$ steps to exit.* Put differently: for all x in this crescent-shaped region $U^\varepsilon(x) \leq \delta/2$. This is clearly sufficient to justify (1.19).

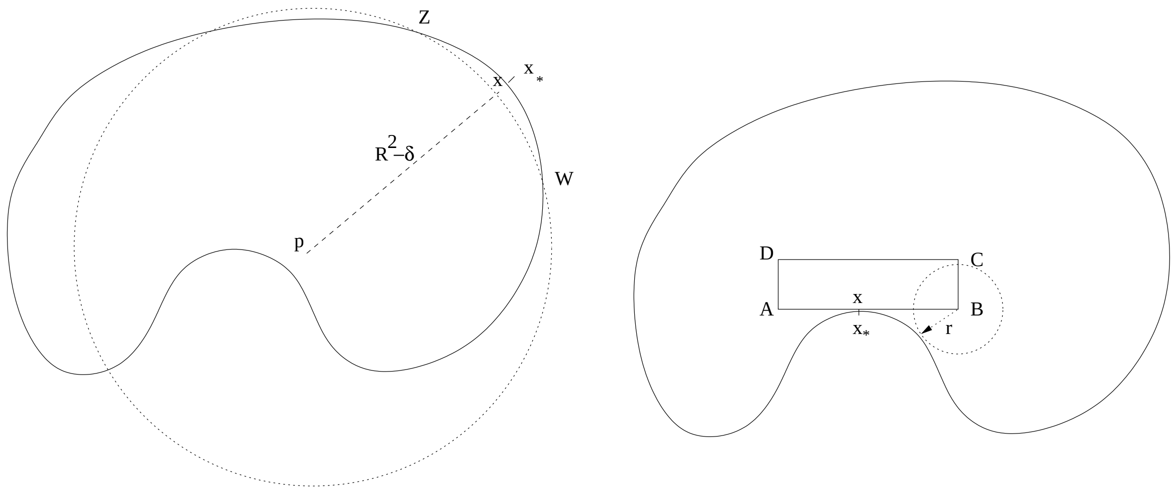

Figure 5: *Left: The crescent-shaped region in the proof of (1.19). Right: the rectangle ABCD in the proof of (1.20).*

The proof of our claim is easy. Consider the following strategy for Paul: *at position x, choose the direction orthogonal to $x-p$.* Starting from $x = x_0$ in the crescent shaped region, Paul's distance from p increases monotonically, regardless of Carol's choices:

$$\|x_k - p\|^2 = \|x_0 - p\|^2 + 2k\varepsilon^2, \quad \text{for } k = 1, 2 \ldots.$$

This argument applies until Paul leaves the crescent-shaped region. Since his distance from p is monotonically increasing, he can only leave this region by reaching $\partial\Omega$. This happens in at most K steps, where $\|x_0 - p\|^2 + 2K\varepsilon^2 = R^2$. Since x_0 was in the crescent-shaped region, $\|x_0 - p\|^2 \geq R^2 - \delta$, and it follows that $K \leq \frac{1}{2}\delta\varepsilon^{-2}$ as asserted. $\square$

Proof of (1.20). We must show that if Paul starts near the strictly concave part of $\partial\Omega$ then his scaled exit time is large. The proof resembles an argument used in Appendix C.3. The plan is simple: we consider a specific strategy for Carol. The strategy is not optimal, but this doesn't matter: it gives the desired lower bound on Paul's minimum exit time.

Fix $x_* \in \partial\Omega$ and assume $\partial\Omega$ is strictly concave and C^2 near x_*. For any $x \in \Omega$ near x_*, we can find a rectangle with with vertices $ABCD$, entirely contained in Ω, such that x is on the segment AB, and the other three sides BC, CD, and DA stay at least distance $r > 0$ away from $\partial\Omega$ (see the right side of Figure 5). Here r is uniform for all x near x_*, since $\partial\Omega$ is C^2 and strictly concave at x_*. We claim that starting from x, Paul needs at least $\frac{1}{2}r^2\varepsilon^{-2}$ steps to exit. Put differently: we claim that $U^\varepsilon(x) \geq r^2/2$. This is obviously sufficient to justify (1.20).

We may assume, without loss of generality, that $[A, B]$ is parallel to the x_1 axis and lies at the bottom of the rectangle (as in the figure). Suppose Carol's strategy is to *keep Paul's x_2 increments greater than or equal to* 0. In other words, if at a given stage Paul's choice of direction is (v_1, v_2), Carol chooses $b = \pm 1$ so that $v_2 b \geq 0$. Clearly no matter what strategy Paul uses, he cannot exit $ABCD$ along the segment AB. So if he ever exits the rectangle, it happens at some point $\overline{x}$ along BC, CD, or DA. When this happens Paul is far from $\partial\Omega$ – specifically, his distance to $\partial\Omega$ is at least r. Thus he must exit from a ball of radius r about $\overline{x}$ before reaching $\partial\Omega$. Now, the optimal exit strategy for a ball is to choose each v in the tangent direction, and starting from the center one reaches radius r in $\frac{1}{2}r^2\varepsilon^{-2}$ steps. Thus if Carol pursues the proposed strategy, then Paul needs at least $\frac{1}{2}r^2\varepsilon^{-2}$ steps to exit from Ω, no matter what strategy he uses. $\square$

6 Higher dimensions

We explained in Section 1.5 how the game must be modified to get motion by mean curvature in $\mathbb{R}^3$:

(a) Paul chooses two orthogonal unit-length directions, i.e. vectors $v, w \in \mathbb{R}^3$ with $\|v\| = \|w\| = 1$ and $v \perp w$.

(b) Carol chooses whether to let Paul's choices stand or reverse them – i.e. she chooses $b = \pm 1$, $\beta = \pm 1$ and replaces v, w with bv and βw.

(c) Paul takes steps of size $\sqrt{2}\varepsilon$ in each direction, moving from x to $x + \sqrt{2}\varepsilon(bv + \beta w)$.

The extension to higher dimensions is obvious: in $\mathbb{R}^n$, Paul chooses $n-1$ orthogonal unit vectors, and Carol can reverse any of them.

Our results connecting the scaled exit time to the positive curvature flow made use of the "$\|v\| \le 1$ game." Its analogue in $\mathbb{R}^3$ replaces (a) above with

(a') Paul chooses two orthogonal vectors with the same length, i.e. $v, w \in \mathbb{R}^3$ with $\|v\| = \|w\| \le 1$ and $v \perp w$.

In dimension n, Paul chooses $n-1$ orthogonal vectors, all of the same norm ≤ 1.

Theorems 2–8 proved convergence of the scaled value functions, for both the time dependent and minimum-exit-time versions of our planar game, and connected the exit time with the positive curvature flow. Their proofs extend straightforwardly to the higher dimensional setting, using Lemma 6.1 below in place of Lemma 2.2. Theorem 1 would extend too, if we knew that the arrival time of the mean-curvature flow was C^3.

Lemma 6.1 *Let ϕ be a C^3 function on compact subset of $\mathbb{R}^3$.*

(a) for any x and any vectors v, w,

$$\text{(6.1)} \qquad \max_{b=\pm 1, \beta=\pm 1} \phi(x + \sqrt{2}\varepsilon(bv + \beta w)) - \phi(x) \ge \varepsilon^2 \left(\langle D^2\phi(x)v, v\rangle + \langle D^2\phi(x)w, w\rangle \right) - C\varepsilon^3.$$

(b) for any x such that $\nabla\phi(x) \neq 0$, we have

$$\text{(6.2)} \qquad \min_{\|v\|=1, \|w\|=1, v\perp w} \max_{b=\pm 1, \beta=\pm 1} \phi(x + \sqrt{2}\varepsilon(bv + \beta w)) - \phi(x) \le \varepsilon^2 \left(\Delta\phi(x) - \langle D^2\phi \frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle \right) + C\varepsilon^3.$$

(c) for any x such that $\nabla\phi(x) \neq 0$ we have

$$\min_{\|v\|=1,\|w\|=1,v\perp w} \max_{b=\pm1,\beta=\pm1} \phi(x+\sqrt{2}\varepsilon(bv+\beta w)) - \phi(x) \geq \varepsilon^2 \left(\Delta\phi(x) - \langle D^2\phi(x)\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle\right) - C\left(1+\frac{1}{|\nabla\phi(x)|}\right)\varepsilon^3. \tag{6.3}$$

In each estimate, the constant C depends only on the C^3 norm of ϕ.

Proof: By Taylor expansion

$$\begin{aligned}
\phi(x+\sqrt{2}\varepsilon(bv+\beta w)) &= \phi(x) + \sqrt{2}\varepsilon\nabla\phi(x)\cdot(bv+\beta w) \\
&\quad + \varepsilon^2\langle D^2\phi(x)(bv+\beta w),(bv+\beta w)\rangle + O(\varepsilon^3) \\
&= \phi(x) + \sqrt{2}\varepsilon\nabla\phi(x)\cdot(bv+\beta w) \\
&\quad + \varepsilon^2\left(\langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle + 2b\beta\langle D^2\phi(x)v,w\rangle\right) + O(\varepsilon^3).
\end{aligned} \tag{6.4}$$

For any given v and w, we can choose $b=\pm1$ so that $b\nabla\phi(x)\cdot v = |\nabla\phi(x)\cdot v|$. For this b

$$\begin{aligned}
\phi(x+\sqrt{2}\varepsilon(bv+\beta w)) - \phi(x) &\geq \sqrt{2}\varepsilon|\nabla\phi(x)\cdot v| + \varepsilon^2\left(\langle D^2\phi(x)v,v\rangle) + \langle D^2\phi(x)w,w\rangle\right) \\
&\quad + \sqrt{2}\beta\varepsilon\nabla\phi(x)\cdot w + 2\varepsilon^2\beta b\langle D^2\phi(x)v,w\rangle + O(\varepsilon^3).
\end{aligned} \tag{6.5}$$

Grouping the terms that are linear in β, we can choose $\beta=\pm1$ to make their net effect positive; therefore

$$\begin{aligned}
\max_{b=\pm1,\beta=\pm1} \phi(x+\sqrt{2}\varepsilon(bv+\beta w)) - \phi(x) &\geq \sqrt{2}\varepsilon|\nabla\phi(x)\cdot v| \\
&\quad + \varepsilon^2\left(\langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle\right) + O(\varepsilon^3).
\end{aligned} \tag{6.6}$$

This proves assertion (a).

Our estimate (6.6) treats v and w asymmetrically. Of course the analogous estimate holds with v replaced by w. Later we will make use of the symmetrized result obtained by averaging these two estimates:

$$\begin{aligned}
\max_{b=\pm1,\beta=\pm1} \phi(x+\sqrt{2}\varepsilon(bv+\beta w)) - \phi(x) &\geq \frac{\sqrt{2}}{2}\varepsilon\left(|\nabla\phi(x)\cdot v| + |\nabla\phi(x)\cdot w|\right) \\
&\quad + \varepsilon^2\left(\langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle\right) + O(\varepsilon^3).
\end{aligned} \tag{6.7}$$

To prove assertion (b), we make a convenient choice of v and w. By hypothesis $\nabla\phi(x) \neq 0$. It therefore makes sense to restrict the quadratic form associated with $D^2\phi(x)$ to the plane perpendicular to $\nabla\phi(x)$. The resulting 2D quadratic form can be diagonalized. We choose v and w to be its (orthonormal) eigenvectors, and we denote the associated eigenvalues by k_1 and k_2. Notice that v, w, and $\nabla\phi(x)$ form an orthonormal basis of $\mathbb{R}^3$. Therefore

$$\begin{aligned}
\Delta\phi(x) &= \langle D^2\phi(x)\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle + k_1 + k_2 \\
&= \langle D^2\phi(x)\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle + \langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle.
\end{aligned} \tag{6.8}$$

Since v and w are perpendicular for $D^2\phi$, in other words $\langle D^2\phi(x)v, w\rangle = 0$, we have, for all $b = \pm 1, \beta = \pm 1$,

$$\begin{aligned}\phi(x+\sqrt{2}\varepsilon(bv+\beta w)) &= \varepsilon^2\left(\langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle\right) + O(\varepsilon^3)\\ &= \varepsilon^2\left(\Delta\phi(x) - \langle D^2\phi(x)\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle\right) + O(\varepsilon^3).\end{aligned}$$

This proves (b).

For assertion (c), we observe (as in Lemma 2.2), the existence of constants $c_1 > 0, C > 0$ such that for all unit vectors v and w with $v \perp w$,

(i) $|\nabla\phi(x)\cdot v| \geq c_1\varepsilon$ or $|\nabla\phi(x)\cdot w| \geq c_1\varepsilon$ imply

$$\begin{aligned}\text{(6.9)}\quad &\frac{\sqrt{2}}{2}\varepsilon\left(|\nabla\phi(x)\cdot v| + |\nabla\phi(x)\cdot w|\right) + \varepsilon^2\left(\langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle\right)\\ &\qquad\geq \varepsilon^2\left(\Delta\phi(x) - \langle D^2\phi(x)\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle\right)\end{aligned}$$

(ii) $|\nabla\phi(x)\cdot v| \leq c_1\varepsilon$ and $|\nabla\phi(x)\cdot w| \leq c_1\varepsilon$ imply

$$\text{(6.10)}\quad \left|\langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle + \langle D^2\phi(x)\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle - \Delta\phi(x)\right| \leq C\left(\frac{\varepsilon}{|\nabla\phi(x)|}\right).$$

(The main point is that when $\nabla\phi\cdot v$ and $\nabla\phi\cdot w$ are both small, v, w and $\nabla\phi(x)/|\nabla\phi(x)|$ are close to being an orthonormal basis of $\mathbb{R}^3$.)

If $|\nabla\phi(x)\cdot v| \geq c_1\varepsilon$ or $|\nabla\phi(x)\cdot w| \geq c_1\varepsilon$ then combining (6.7) with (6.9), we obtain the desired result. If on the other hand $|\nabla\phi(x)\cdot v| \leq c_1\varepsilon$ and $|\nabla\phi(x)\cdot w| \leq c_1\varepsilon$, then (6.7) and (6.10) give

$$\begin{aligned}\text{(6.11)}\quad &\max_{b=\pm1,\beta=\pm1}\phi(x+\sqrt{2}\varepsilon(bv+\beta w)) - \phi(x) \geq \varepsilon^2\left(\langle D^2\phi(x)v,v\rangle + \langle D^2\phi(x)w,w\rangle\right) + O(\varepsilon^3)\\ &\qquad\geq \varepsilon^2\Big(\Delta\phi(x) - \langle D^2\phi(x)\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\rangle\Big) - C\left(1 + \frac{1}{|\nabla\phi(x)|}\right)\varepsilon^3.\end{aligned}$$

□

APPENDICES

A Addendum to Section 3: Regularity of the exit time

This appendix proves Proposition 3.1, the regularity result needed for the verification argument in Section 3. Our focus is thus on the evolution of a closed, strictly convex, planar curve $\partial\Omega$ as it evolves under motion by curvature. It remains convex and becomes asymptotically circular as it shrinks to a point at the extinction time $T = \frac{1}{2\pi}|\Omega|$ [GH]. We write x_* for the point to which it shrinks (which has no simple formula in terms of $\partial\Omega$).

Since the evolving curve is convex, it can be parametrized by its tangent angle θ. Thus it is the locus of

$$\gamma(\theta, t) = (x(\theta, t), y(\theta, t))$$

and its unit tangent and (inward) normal vectors are

$$\mathbf{t}(\theta, t) = (\cos\theta, \sin\theta), \quad \mathbf{n}(\theta, t) = (-\sin\theta, \cos\theta).$$

We are interested in $U(x)$ = the time when the moving boundary arrives at x. It is characterized by

$$U(\gamma(\theta, t)) = t. \tag{A.1}$$

Its smoothness is obvious away from x_*; our task is to show that U is C^3 at x_*, with

$$D^2U(x_*) = -I \quad \text{and} \quad D^3U(x_*) = 0. \tag{A.2}$$

We shall accomplish this task by differentiating (A.1) several times, then considering the limit $t \to T$.

The smoothness of U near x_* is clearly related to the roundness of γ as it shrinks to a point. The estimates expressing this roundness are best expressed in terms of

$$k(\theta, t) = \text{curvature of } \gamma \text{ at angle } \theta \text{ and time } t$$

or, even better, the rescaled curvature as a function of logarithmically stretched time:

$$\kappa(\theta, \tau) = \sqrt{2(T-t)}\,k(\theta, t), \quad \text{where} \quad \tau = -\frac{1}{2}\log(T-t). \tag{A.3}$$

The fact that γ undergoes motion by curvature is equivalent to the PDE

$$k_t = k^2 k_{\theta\theta} + k^3, \tag{A.4}$$

and also to the PDE

$$\kappa_\tau = \kappa^2\kappa_{\theta\theta} + \kappa^3 - \kappa. \tag{A.5}$$

The asymptotic roundness of γ was proved by Gage and Hamilton, and is expressed by the following estimates:

$$\begin{cases} \kappa \to 1 \text{ uniformly as } \tau \to \infty \\ \|\partial^\ell\kappa/\partial\theta^\ell\|_\infty \le C_{\alpha,\ell}e^{-2\alpha\tau} \text{ for all } 0 < \alpha < 1 \text{ and } \ell \ge 1 \\ \|\partial^\ell k/\partial\theta^\ell\|_\infty \le C_{\alpha,\ell}(T-t)^{\alpha-1/2} \text{ for all } 0 < \alpha < 1 \text{ and } \ell \ge 1. \end{cases} \tag{A.6}$$

These assertions are respectively Corollary 5.6, Theorem 5.7.1, and Corollary 5.7.2 of [GH].

The estimates (A.6) are enough to prove U is C^2. For the proof that it is C^3 however we will need the following sharpened version of the first assertion:

Lemma A.1 $\|\kappa - 1\|_\infty \leq C_\alpha e^{-2\alpha\tau}$ *for all* $0 < \alpha < 1$.

Proof: The essential idea is that $\kappa = 1$ is an *unstable* state for the PDE (A.5). If κ ever wandered far from 1 the linear instability would take over; but we know that $\kappa \to 1$ uniformly, so it must in fact stay very close to 1.

To pursue this idea, let $\kappa = 1 + \delta$. The PDE becomes

$$\delta_\tau = 2\delta + [(1+\delta)^2\delta_{\theta\theta} + 3\delta^2 + \delta^3].$$

We know $\delta \to 0$ and $|\delta_{\theta\theta}| \leq Ce^{-2\alpha\tau}$, so this gives (at any fixed θ)

$$|\delta|_\tau \leq (2+\varepsilon)|\delta| + Ce^{-2\alpha\tau}$$

and

$$|\delta|_\tau \geq (2-\varepsilon)|\delta| - Ce^{-2\alpha\tau}$$

for any $\varepsilon > 0$ and sufficiently large τ. Now observe that the differential inequality

$$f_\tau \leq \beta f + Ce^{-2\alpha\tau}$$

integrates to

$$e^{-\beta\tau_1} f(\tau_1) - e^{-\beta\tau_0} f(\tau_0) \leq \frac{C}{2\alpha+\beta}\left[e^{-(2\alpha+\beta)\tau_0} - e^{-(2\alpha+\beta)\tau_1}\right].$$

If in addition f is uniformly bounded then taking $\tau_1 \to \infty$ gives

$$f(\tau_0) \geq -\frac{C}{2\alpha+\beta}e^{-2\alpha\tau_0}.$$

A similar argument starting from $f_\tau \geq \beta f - Ce^{-2\alpha\tau}$ gives

$$f(\tau_0) \leq \frac{C}{2\alpha+\beta}e^{-2\alpha\tau_0}.$$

Applying these results to $f(\tau) = |\delta(\theta,\tau)|$ we get

$$|\kappa - 1| \leq Ce^{-2\alpha\tau}$$

as desired. $\square$

We need one more basic fact: the decomposition of γ_t and γ_θ with respect to the tangent $\mathbf{t}$ and normal $\mathbf{n}$ at angle θ and time t. We claim that

$$\gamma_\theta = \frac{1}{k}\mathbf{t} \quad \text{and} \quad \gamma_t = k\mathbf{n} - k_\theta\mathbf{t}. \tag{A.7}$$

The first assertion is easy: if s denotes arclength along the curve then

$$\gamma_\theta = \frac{ds}{d\theta}\gamma_s = \frac{1}{k}\mathbf{t}$$

since $k = d\theta/ds$. To justify the second half of (A.7), we observe that $\langle \gamma_t, \mathbf{n} \rangle = k$, since the curve is moving with normal velocity k. Differentiating this in θ gives

$$\langle \gamma_{t\theta}, \mathbf{n} \rangle - \langle \gamma_t, \mathbf{t} \rangle = k_\theta$$

since $\mathbf{t} = (\cos\theta, \sin\theta)$ and $\mathbf{n} = (-\sin\theta, \cos\theta)$. But

$$\langle \gamma_{t\theta}, \mathbf{n} \rangle = \frac{\partial}{\partial t} \langle \gamma_\theta, \mathbf{n} \rangle = 0$$

by (A.7). So $\langle \gamma_t, \mathbf{t} \rangle = -k_\theta$, as asserted.

Proof of Lemma 3.1: Differentiating (A.1) using the chain rule gives

$$DU(\gamma) \cdot \gamma_t = 1 \quad \text{and} \quad DU(\gamma) \cdot \gamma_\theta = 0. \tag{A.8}$$

Using (A.7) we conclude that

$$\nabla U = \frac{1}{k} \mathbf{n}. \tag{A.9}$$

(This relation is also an immediate consequence of the definition of U as the arrival time of the curvature flow.)

Differentiating (A.8) gives

$$\begin{aligned} \langle D^2U(\gamma), \gamma_t \otimes \gamma_t \rangle + DU(\gamma) \cdot \gamma_{tt} &= 0 \\ \langle D^2U(\gamma), \gamma_t \otimes \gamma_\theta \rangle + DU(\gamma) \cdot \gamma_{\theta t} &= 0 \\ \langle D^2U(\gamma), \gamma_\theta \otimes \gamma_\theta \rangle + DU(\gamma) \cdot \gamma_{\theta\theta} &= 0. \end{aligned} \tag{A.10}$$

To simplify these, observe that

$$\begin{aligned} \gamma_{tt} &= k_t \mathbf{n} - k_{\theta t} \mathbf{t} \\ \gamma_{\theta t} &= -k^{-2} k_t \mathbf{t} \\ \gamma_{\theta\theta} &= \frac{1}{k} \mathbf{n} - \frac{k_\theta}{k^2} \mathbf{t} \end{aligned} \tag{A.11}$$

so we may write (A.10) as

$$\begin{aligned} \langle D^2U(\gamma), \gamma_t \otimes \gamma_t \rangle + \frac{k_t}{k} &= 0 \\ \langle D^2U(\gamma), \gamma_t \otimes \gamma_\theta \rangle &= 0 \\ \langle D^2U(\gamma), \gamma_\theta \otimes \gamma_\theta \rangle + \frac{1}{k^2} &= 0. \end{aligned} \tag{A.12}$$

The C^2 character of U follows easily. Indeed, the frame

$$\begin{aligned} \gamma_t / k &= \mathbf{n} - \frac{k_\theta}{k} \mathbf{t} \\ k \gamma_\theta &= \mathbf{t} \end{aligned}$$

is nearly orthonormal as $t \to T$, since

$$\frac{k_\theta}{k} = \frac{\kappa_\theta}{\kappa} \to 0$$

by (A.6). The components of D^2U in this frame are

$$\begin{aligned} \langle D^2U(\gamma), \frac{\gamma_t}{k} \otimes \frac{\gamma_t}{k} \rangle &= -\frac{k_t}{k^3} \\ \langle D^2U(\gamma), \frac{\gamma_t}{k} \otimes k\gamma_\theta \rangle &= 0 \\ \langle D^2U(\gamma), k\gamma_\theta \otimes k\gamma_\theta \rangle &= -1. \end{aligned} \tag{A.13}$$

We claim that

$$\frac{k_t}{k^3} \to 1 \text{ as } t \to T. \tag{A.14}$$

Indeed, from the definition (A.3) we have $k(\theta, t) = \rho(t)\kappa(\theta, \tau)$ with $\rho = 1/\sqrt{2(T-t)}$ and $\tau = -\frac{1}{2}\log(T-t)$. Since $\rho_t = \rho^3$ and $d\tau/dt = \rho^2$ we have

$$\frac{k_t}{k^3} = \frac{\rho^3\kappa + \rho\kappa_t}{\rho^3\kappa^3} = \frac{1}{\kappa^2} + \frac{\kappa_\tau}{\kappa^3}.$$

We conclude using (A.5) and (A.6) that (A.14) holds. Thus $D^2U(x)$ has a limit as x approaches x_*, and

$$D^2U(x_*) = -I.$$

The proof that U is C^3 at x_* is similar in concept, though more complicated in detail. We start by differentiating (A.12), to get

$$\langle D^3U(\gamma), \gamma_t \otimes \gamma_t \otimes \gamma_t \rangle + 2\langle D^2U(\gamma), \gamma_{tt} \otimes \gamma_t \rangle + \partial_t \left(\frac{k_t}{k}\right) = 0 \tag{A.15}$$

$$\langle D^3U(\gamma), \gamma_\theta \otimes \gamma_t \otimes \gamma_t \rangle + \langle D^2U(\gamma), \gamma_{tt} \otimes \gamma_\theta \rangle + \langle D^2U(\gamma), \gamma_t \otimes \gamma_{t\theta} \rangle = 0 \tag{A.16}$$

$$\langle D^3U(\gamma), \gamma_\theta \otimes \gamma_\theta \otimes \gamma_t \rangle + \langle D^2U(\gamma), \gamma_{t\theta} \otimes \gamma_\theta \rangle + \langle D^2U(\gamma), \gamma_t \otimes \gamma_{\theta\theta} \rangle = 0 \tag{A.17}$$

$$\langle D^3U(\gamma), \gamma_\theta \otimes \gamma_\theta \otimes \gamma_\theta \rangle + 2\langle D^2U(\gamma), \gamma_{\theta\theta} \otimes \gamma_\theta \rangle + \partial_\theta \left(\frac{1}{k^2}\right) = 0 \tag{A.18}$$

The terms involving γ_{tt}, $\gamma_{t\theta}$, and $\gamma_{\theta\theta}$ can be simplified by expressing these vectors in the frame γ_t, γ_n then using (A.12). An elementary calculation using (A.4), (A.7) and (A.11) gives

$$\begin{aligned} \gamma_{tt} &= \frac{k_t}{k}\gamma_t + (k_t k_\theta - k k_{\theta t})\gamma_\theta \\ \gamma_{\theta t} &= -k(k_{\theta\theta} + k)\gamma_\theta \\ \gamma_{\theta\theta} &= \frac{1}{k^2}\gamma_t, \end{aligned}$$

and, using these relations, (A.15)-(A.18) simplify to

$$\text{(A.19)} \qquad \langle D^3U, \frac{\gamma_t}{k}\otimes\frac{\gamma_t}{k}\otimes\frac{\gamma_t}{k}\rangle = \frac{1}{k^3}\left[2\left(\frac{k_t}{k}\right)^2 - \partial_t\left(\frac{k_t}{k}\right)\right]$$

$$\text{(A.20)} \qquad \langle D^3U, k\gamma_\theta\otimes\frac{\gamma_t}{k}\otimes\frac{\gamma_t}{k}\rangle = \frac{1}{k^3}(k_tk_\theta - kk_{\theta t})$$

$$\text{(A.21)} \qquad \langle D^3U, k\gamma_\theta\otimes k\gamma_\theta\otimes\frac{\gamma_t}{k}\rangle = 0$$

$$\text{(A.22)} \qquad \langle D^3U, k\gamma_\theta\otimes k\gamma_\theta\otimes k\gamma_\theta\rangle = -k^3\partial_\theta\left(\frac{1}{k^2}\right).$$

Our goal is to show that D^3U is C^3 at x_*, with $D^3U(x_*) = 0$. We shall achieve this by showing that the right hand sides of (A.19)-(A.22) tend to 0 as $t \to T$. Easiest first: the right side of (A.22) is

$$-k^3\partial_\theta\left(\frac{1}{k^2}\right) = 2k_\theta \to 0$$

by (A.6). The right side of (A.20) is only a little more difficult: by (A.4)

$$k_tk_\theta - kk_{\theta t} = k_\theta(k^2k_{\theta\theta} + k^3) - k(k^2k_{\theta\theta} + k^3)_\theta$$

so

$$\frac{1}{k^3}(k_tk_\theta - kk_{\theta t}) = -\frac{k_\theta}{k}k_{\theta\theta} - 2k_\theta - k_{\theta\theta\theta};$$

each of the terms on the right tends to 0, using (A.6) and the fact that $k_\theta/k = \kappa_\theta/\kappa$.

It remains to show that the right side of (A.19) tends to 0 as $t \to T$. This is a bit more sensitive. Recall that $k(\theta,t) = \rho(t)\kappa(\theta,\tau)$ with $\rho = 1/\sqrt{2(T-t)}$ and $\tau = -\frac{1}{2}\log(T-t)$. We have

$$k_t = \rho_t\kappa + \rho\kappa_t = \rho^3(\kappa + \kappa_\tau)$$

and similarly

$$k_{tt} = \rho^5(3\kappa + 4\kappa_\tau + \kappa_{\tau\tau}).$$

Therefore

$$\text{(A.23)} \qquad \frac{2\left(\frac{k_t}{k}\right)^2 - \partial_t\left(\frac{k_t}{k}\right)}{k^3} = \frac{3k_t^2 - kk_{tt}}{k^5} = \frac{2\rho\kappa\kappa_\tau + 3\rho\kappa_\tau^2 - \rho\kappa\kappa_{\tau\tau}}{\kappa^5}.$$

To see that the first two terms in the numerator tend to 0 we combine (A.5) with (A.6) and Lemma A.1:

$$\text{(A.24)} \qquad |\kappa_\tau| = |\kappa^2\kappa_{\theta\theta} + \kappa^3 - \kappa| \le Ce^{-2\alpha\tau}$$

for any $\alpha < 1$. Since $\rho = (1/\sqrt{2})e^\tau$, it follows that $\rho\kappa_\tau \to 0$ and $\rho\kappa_\tau^2 \to 0$, as asserted. The argument for the third term in the numerator of (A.23) is similar: differentiating the κ-equation with respect to τ gives

$$\begin{aligned}\kappa_{\tau\tau} &= 2\kappa\kappa_{\theta\theta}\kappa_\tau + \kappa^2\kappa_{\theta\theta\tau} + 3\kappa^2\kappa_\tau - \kappa_\tau\\ &= \kappa_\tau(2\kappa\kappa_{\theta\theta} + 3\kappa^2 - 1) + \kappa^2(\kappa^2\kappa_{\theta\theta} + \kappa^3 - \kappa)_{\theta\theta}.\end{aligned}$$

It follows from (A.6) and (A.24) that

$$|\kappa_{\tau\tau}| \leq Ce^{-2\alpha\tau}$$

for any $\alpha < 1$, whence $\rho\kappa_{\tau\tau} \to 0$. Thus U is C^3 and $D^3U(x_*) = 0$, as asserted. $\square$.

Remark: It is natural to conjecture that the analogous result holds in any dimension. Thus, if Ω is a smoothly bounded, strictly convex domain in $\mathbb{R}^n$ with $n > 2$, we conjecture that the arrival time $U(x)$ associated with the mean curvature flow is C^3, with $D^3U(x_*) = 0$. Huisken showed in [H2] that U is C^2, with $D^2U(x_*) = -\frac{1}{n-1}I$. The proof takes just a few lines, given the higher-dimensional analogues of (A.6) which were proved in [H1]. However a C^3 estimate seems to require a different technique – including, most likely, a higher-dimensional analogue of Lemma A.1.

B Addendum to Section 4: Equicontinuity of the value functions

Section 3 proved the existence of the limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ and characterized it simultaneously, using viscosity methods. Briefly: we showed that $\overline{u}$ and $\underline{u}$ are respectively a subsolution and supersolution of the level-set equation for motion by curvature. Then a standard comparison theorem gave $\underline{u} = \overline{u}$, implying existence of the limit.

However it is also possible to prove directly, by elementary means, that the family $\{u^\varepsilon\}$ is compact (which shows that the limit exists, at least for a subsequence). We think the argument is interesting, because it captures the sense in which our game is a stable discretization of motion by curvature.

This appendix discusses only the time-dependent version of the game. A similar proof of equicontinuity can be given for the exit-time game in a strictly convex domain. However we do not know how to prove equicontinuity for the exit-time game in a nonconvex domain. This is related to the question whether the limiting function U can be discontinuous.

Proposition B.1 *Consider the time-dependent version of the game, as in Section 4. Assume the objective function u_0 is C^2. Then the associated value functions $u^\varepsilon(x,t)$ are uniformly equicontinuous.*

Proof: The argument rests on two lemmas:

Lemma B.1 *The process of stepping backward in time is L^∞-stable. In other words, if*

$$f_1(x) = \min_{\|v\|=1} \max_{b=\pm 1} f_0(x + \sqrt{2}\varepsilon bv) \tag{B.1}$$

and

$$\tilde{f}_1(x) = \min_{\|v\|=1} \max_{b=\pm 1} \tilde{f}_0(x + \sqrt{2}\varepsilon bv) \tag{B.2}$$

then

$$\|f_1 - \tilde{f}_1\|_{L^\infty} \leq \|f_0 - \tilde{f}_0\|_{L^\infty}. \tag{B.3}$$

This relation holds for any pair of continuous functions $f_0, \tilde{f}_0$, and any $\varepsilon > 0$.

Lemma B.2 *If f_0 has uniformly bounded second derivatives then we can control how much the value changes at the first backward timestep. In fact, the function f_1 defined by (B.1) satisfies*

$$\|f_1 - f_0\|_{L^\infty} \le M\varepsilon^2, \tag{B.4}$$

for some constant M, which depends on f_0 but not on ε.

Let's show right away that Lemmas B.1 and B.2 imply equicontinuity of the family $\{u^\varepsilon(x,t)\}$ as $\varepsilon \to 0$. First, we show that if u_0 is uniformly Lipschitz in x with Lipschitz constant C, then the same is true of $u^\varepsilon(x,t)$ at each discrete time:

$$|u^\varepsilon(x, T - k\varepsilon^2) - u^\varepsilon(x', T - k\varepsilon^2)| \le C|x - x'| \tag{B.5}$$

for every ε, every $k \in \mathbb{Z}$, and every x, x'. The proof is by induction on k. The assertion is true for $k = 0$ by the definition of C, since $u_\varepsilon(x,T) = u_0(x)$. For the inductive step we may assume that (B.5) is true at time $T - k\varepsilon^2$, and we must prove that it holds at time $T - (k+1)\varepsilon^2$. For any $a \in \mathbb{R}^2$ we may apply Lemma B.1 with $f_0(x) = u^\varepsilon(x, T - k\varepsilon^2)$ and $\tilde{f}_0(x) = u^\varepsilon(x - a, T - k\varepsilon^2)$ (these functions are continuous, by the inductive hypothesis). The associated $f_1(x)$ and $\tilde{f}_1(x)$ are $u^\varepsilon(x, T-(k+1)\varepsilon^2)$ and $u^\varepsilon(x - a, T - (k+1)\varepsilon^2)$ respectively. Combining the conclusion of Lemma B.1 with the inductive hypothesis, we get

$$\begin{aligned} &|u^\varepsilon(x - a, T - (k+1)\varepsilon^2) - u^\varepsilon(x, T - (k+1)\varepsilon^2)| \\ &\qquad \le \|u^\varepsilon(x - a, T - k\varepsilon^2) - u^\varepsilon(x, T - k\varepsilon^2)\|_{L^\infty} \le C|a|. \end{aligned} \tag{B.6}$$

Taking $a = x - x'$ this gives (B.5) at time $T - (k+1)\varepsilon^2$. The induction is now complete.

Next, we show that if u_0 has uniformly-bounded second derivatives then u^ε is "Lipschitz continuous in discrete time," i.e.

$$|u^\varepsilon(x,t) - u^\varepsilon(x,t')| \le M|t - t'| \quad \text{for all } x \text{ and all } t = T - k\varepsilon^2,\ t' = T - k'\varepsilon^2. \tag{B.7}$$

This follows from the lemmas, combined with the autonomous character of our game and an inductive argument. Indeed, for any $t = T - k\varepsilon^2$,

$$u^\varepsilon(x, t + \varepsilon^2) = \begin{cases} \text{optimal value at position } x \text{ and time } t, \text{ if the} \\ \text{final time is } T - \varepsilon^2 \text{ and the objective is } u_0(x) \end{cases}$$

while

$$u^\varepsilon(x,t) = \begin{cases} \text{optimal value at position } x \text{ and time } t, \text{ if the} \\ \text{final time is } T - \varepsilon^2 \text{ and the objective is } \tilde{u}_0(x) = u^\varepsilon(x, T - \varepsilon^2) \end{cases}$$

By Lemma B.2, the two objectives don't differ very much:

$$\|u_0 - \tilde{u}_0\|_{L^\infty} \le M\varepsilon^2.$$

By an obvious induction based on Lemma B.1, it follows that the two values don't differ very much either:

$$|u^\varepsilon(x, t + \varepsilon^2) - u^\varepsilon(x,t)| \le M\varepsilon^2 \quad \text{for any } t = T - k\varepsilon^2. \tag{B.8}$$

The desired Lipschitz continuity (B.7) follows easily, by applying (B.8) to each timestep between t and t'.

Finally, we observe that Lipschitz continuity in space (B.5) and "Lipschitz continuity in discrete time" (B.7) imply equicontinuity. Indeed, we may consider $u^\varepsilon(x,t)$ to be defined at all times $t < T$ by linear interpolation. Thus extended, the functions $\{u^\varepsilon\}$ are Lipschitz continuous in the conventional sense, uniformly in ε.

Proof of Lemma B.1: Let v_1 and b_1 be optimal in the definition of u_1, so that

$$f_1(x) = f_0(x + \sqrt{2}\varepsilon b_1 v_1) \geq f_0(x - \sqrt{2}\varepsilon b_1 v_1) \tag{B.9}$$

with $\|v_1\| = 1$ and $b_1 = \pm 1$. Using v_1 as a trial-vector in the definition of $\tilde{f}_1$, we have (for some $\tilde{b}_1 = \pm 1$)

$$\tilde{f}_1(x) \leq \tilde{f}_0(x + \sqrt{2}\varepsilon \tilde{b}_1 v_1) \leq f_0(x + \sqrt{2}\varepsilon \tilde{b}_1 v_1) + \|f_0 - \tilde{f}_0\|_{L^\infty}. \tag{B.10}$$

From (B.9) we have $f_1(x) \geq f_0(x + \sqrt{2}\varepsilon \tilde{b}_1 v_1)$; subtracting this from (B.10) we conclude that

$$\tilde{f}_1(x) - f_1(x) \leq \|f_0 - \tilde{f}_0\|_{L^\infty}.$$

A symmetric argument gives

$$f_1(x) - \tilde{f}_1(x) \leq \|f_0 - \tilde{f}_0\|_{L^\infty};$$

together, these amount to the desired inequality (B.3). □

Proof of Lemma B.2: By Taylor expansion we have

$$\max_{b=\pm 1} f_0(y + \sqrt{2}\varepsilon b v) = f_0(y) + \sqrt{2}\varepsilon |v \cdot \nabla f_0(y)| + \varepsilon^2 \langle D^2 f_0(y) v, v\rangle + o(\varepsilon^2) \tag{B.11}$$

for any unit vector v. Choosing $v \perp \nabla f_0(y)$ we conclude that

$$\min_{\|v\|=1} \max_{b=\pm 1} f_0(y + \sqrt{2}\varepsilon b v) \leq f_0(y) + M\varepsilon^2.$$

On the other hand, (B.11) implies

$$\max_{b=\pm 1} f_0(y + \sqrt{2}\varepsilon b v) \geq f_0(y) - M\varepsilon^2$$

for any v, so

$$\min_{\|v\|=1} \max_{b=\pm 1} f_0(y + \sqrt{2}\varepsilon b v) \geq f_0(y) - M\varepsilon^2.$$

□

C Addendum to Section 5: A weak comparison theorem and an example, by Guy Barles and Francesca Da Lio

This Appendix proves Theorem 4, the comparison result needed in Section 5 for the boundary value problem which characterizes Paul's arrival time. It also gives an example, showing that $\underline{U}$ and $\overline{U}$ can be different.

C.1 A weak comparison result

We shall prove a "weak comparison result" result for the following problem

$$\text{(C.1)}\qquad \begin{cases} -\left(\Delta u - \dfrac{\langle D^2uDu, Du\rangle}{|Du|^2}\right) - 1 &= 0 \quad \text{in } \Omega \\ u(x) &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

By contrast with a "strong comparison result" which says that any usc subsolution is below any lsc supersolution and which implies, as a by-product, the continuity of the (unique) solution, our "weak comparison result" yields the same type of result but by using suitable envelopes of the sub and supersolutions. It implies the uniqueness of the solution (in a suitable sense) even in the case when it is discontinuous. In the next section, we provide an example of such a case for (C.1). This uses the starshapedness of Ω in a crucial way. Starshapedness is a convenient tool to get comparison results for discontinuous solutions (see for example [Gi2] for another example of such a use though in a quite different setting).

For simplicity of notation we write the equation below as $F(Du, D^2u) = 1$, where, for $p \neq 0$ and $M \in S^N$, F is given by

$$F(p, M) = -\left(\mathrm{Tr}[(Id - \frac{p \otimes p}{|p|^2})M]\right) .$$

If $p = 0$, the definition of viscosity sub and supersolutions use suitable extensions by upper or lower semicontinuity.

The basic assumption on the domain Ω is the following

(H0) Ω is a bounded open subset of $I\!\!R^N$ which is starshaped with respect to the origin, i.e. there exists $\lambda_0 > 1$ and $\gamma > 0$ such that, for all $\lambda \in (1, \lambda_0)$

$$\text{(C.2)}\qquad \mathrm{dist}(x, \partial\Omega) \geq \gamma(1 - \lambda^{-1}) \ \text{ if } x \in \lambda^{-1}\overline{\Omega}$$

We write $BUSC(\overline{\Omega})$ (respectively, $BLSC(\overline{\Omega})$) for the class of bounded upper semicontinuous (respectively, bounded lower semicontinuous) functions on $\overline{\Omega}$. Here is a more careful statement of Theorem 4.

Theorem 4 *Assume* **(H0)**. *Let $u \in BUSC(\overline{\Omega}), v \in BLSC(\overline{\Omega})$ be respectively viscosity sub- and supersolutions of (C.1). Then $u_*(x) \leq v(x)$ and $u(x) \leq v^*(x)$ in $\overline{\Omega}$.*

Proof: We start by two key remarks.

1. We first claim that every $v \in BLSC(\overline{\Omega})$ viscosity supersolution of (C.1) is nonnegative. Indeed, since v is lsc on $\overline{\Omega}$, it achieves its minimum at some point $\bar{x}$. By applying the definition of viscosity supersolution at this point, it is easy to see that the viscosity inequality associated to the equation cannot hold and therefore necessarily, $\bar{x} \in \partial\Omega$ and $v(\bar{x}) \geq 0$. And the claim follows.

2. If $u \in BUSC(\overline{\Omega})$ is a viscosity subsolution of (C.1), then, for all $\delta < 1$, the function $u_\delta(x) = \delta u(x)$ satisfies

$$\text{(C.3)}\qquad F(Du_\delta, D^2u_\delta) \leq \delta < 1 \quad \text{in } \Omega\,.$$

Thus we may assume without restriction that we deal with a strict subsolution of (C.1). In the following, we drop the δ and just denote by u one of the u_δ.

3. The next arguments are inspired from Theorem 3.1 of [BD]. For $\lambda > 1$, close to 1, we set $\Omega_\lambda = \lambda^{-1}\Omega$ and we introduce the function

$$u_\lambda(x) := \lambda^{-2}u(\lambda x),\ x \in \overline{\Omega}_\lambda.$$

Because of the form of the equation, the function u_λ is a subsolution of

$$F(Du_\lambda, D^2u_\lambda) \leq \delta < 1 \quad \text{in } \Omega_\lambda\,. \tag{C.4}$$

In order to prove the result, we are going to show that, if λ is close enough to 1, we have

$$u_\lambda(x) \leq v(x) \quad \text{in } \overline{\Omega}_{\lambda_n} \cap \overline{\Omega}\,. \tag{C.5}$$

Indeed, if $x \in \overline{\Omega}$, we have $\limsup_\lambda u_\lambda(x) \geq u_*(x)$ and (C.5) implies $u_*(x) \leq v(x)$. The other inequality is obtained in a similar way.

4. To prove (C.5), we suppose by contradiction that there exists a sequence $(\lambda_n)_n$, $\lambda_n > 1$, $\lambda_n \to 1$, for which

$$\max_{(\overline{\Omega}_{\lambda_n} \cap \overline{\Omega})} [u_{\lambda_n}(x) - v(x)] > 0\,. \tag{C.6}$$

From now on, we drop the index n to simplify the notations. For fixed $\lambda > 1$ and for all $\varepsilon > 0$, we consider the function $\Phi_\varepsilon \colon \overline{\Omega}_\lambda \times \overline{\Omega} \to I\!R$, defined by

$$\Phi_\varepsilon(x, y) = u_\lambda(x) - v(y) - \frac{|x - y|^4}{\varepsilon^4}.$$

Let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega}_\lambda \times \overline{\Omega}$ be a maximum point of Φ_ε. The inequality $\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \Phi_\varepsilon(x, x)$ for any $x \in \overline{\Omega}_\lambda \cap \overline{\Omega}$ together with the boundedness of u_λ and v yield

$$\frac{|x - y|^4}{\varepsilon^4} \leq C$$

for some constant C depending on $||u_\lambda||_\infty$ and $||v||_\infty$. Moreover by the compactness of $\overline{\Omega}_\lambda \cap \overline{\Omega}$ there exists a subsequence of $(x_\varepsilon, y_\varepsilon)$ (that we continue to denote by $(x_\varepsilon, y_\varepsilon)$) such that $x_\varepsilon, y_\varepsilon$ converge to $x_\lambda \in \overline{\Omega}_\lambda \cap \overline{\Omega}$ as $\varepsilon \to 0$. The following inequalities hold:

$$\liminf_{\varepsilon \to 0} \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \max_{(\overline{\Omega}_\lambda \cap \overline{\Omega})} [u_\lambda(x) - v(x)]\,, \tag{C.7}$$

$$\begin{aligned} \limsup_{\varepsilon \to 0} \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) &\leq \limsup_{\varepsilon \to 0} (u_\lambda(x_\varepsilon) - v(y_\varepsilon)) - \liminf_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^4}{\varepsilon^4} \\ &\leq \max_{(\overline{\Omega}_\lambda \cap \overline{\Omega})} [u_\lambda(x) - v(x)]. \end{aligned} \tag{C.8}$$

By combining (C.7) and (C.8) we get (up to subsequence)

$$|x_\varepsilon - y_\varepsilon| = o(\varepsilon) \ \text{ as } \varepsilon \to 0,$$

$$\lim_{\varepsilon\to 0}[u_\lambda(x_\varepsilon)-v(y_\varepsilon)] = u_\lambda(x_\lambda)-v(x_\lambda) = \max_{(\overline{\Omega}_\lambda\cap\overline{\Omega})}[u_\lambda(x)-v(x)],$$

and the upper and lower semicontinuity of u_λ and v imply that

$$u_\lambda(x_\varepsilon)\to u(x_\lambda),\ v(y_\varepsilon)\to v(x_\lambda), \quad \text{as } \varepsilon\to 0.$$

We claim that, for $\varepsilon>0$ small enough, the viscosity inequalities associated to F hold for both $u_\lambda(x_\varepsilon)$ and $v(y_\varepsilon)$. Indeed, suppose that we have $x_\varepsilon\in\partial\Omega_\lambda$. From (C.6), it follows that we have $u_\lambda(x_\varepsilon)-v(y_\varepsilon)>0$; but since $v(y_\varepsilon)\geq 0$, this implies that $u_\lambda(x_\varepsilon)>0$. Therefore, even if $x_\varepsilon\in\partial\Omega_\lambda$, the boundary condition in the viscosity sense reduces to the F inequality. As a consequence, the F–viscosity inequality holds for $u_\lambda(x_\varepsilon)$ wherever x_ε lies.

On the other hand, since $|x_\varepsilon-y_\varepsilon|=o(\varepsilon)$ and **(H0)** holds, we have that necessarily $y_\varepsilon\in\Omega$ and thus the viscosity inequality hold for $v(y_\varepsilon)$ as well.

5. We denote by $\zeta(x,y):=\dfrac{|x-y|^4}{\varepsilon^4}$. Since the equation is singular at $Du=0$, we have to consider separately the two cases $D\zeta(x_\varepsilon,y_\varepsilon)\neq 0$ and $D\zeta(x_\varepsilon,y_\varepsilon)=0$.

We first assume that there exists a subsequence of $(x_\varepsilon,y_\varepsilon)$ (which we continue to denote by $(x_\varepsilon,y_\varepsilon)$) such that $D\zeta(x_\varepsilon,y_\varepsilon)\neq 0$. By standard arguments (cf. Theorem 3.3 in the "User's guide" of Crandall, Ishii and Lions [CIL]), for any $\alpha>0$, there exists $(p,X)\in\overline{\mathcal{J}}^{2,+}u_\lambda(x_\varepsilon),\ (q,Y)\in\overline{\mathcal{J}}^{2,-}v(y_\varepsilon)$ such that

$$p=q=D_x\zeta(x_\varepsilon,y_\varepsilon)=-D_y\zeta(x_\varepsilon,y_\varepsilon)\ ,$$

and, if $A_\varepsilon=D^2\zeta(x_\varepsilon,y_\varepsilon)$

$$\text{(C.9)}\qquad -(\frac{1}{\alpha}+||A_\varepsilon||)Id\leq\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\leq(Id+\alpha A_\varepsilon)A_\varepsilon\ .$$

Moreover the viscosity inequalities for u_λ and v read

$$\text{(C.10)}\qquad F(p,X)\leq\delta<1\,,$$

$$\text{(C.11)}\qquad F(q,Y)\geq 1\,.$$

From (C.9), it follows that, for any $r,s\in I\!R^N$,

$$\langle Xr,r\rangle-\langle Ys,s\rangle\leq\langle B_\varepsilon(r-s),(r-s)\rangle\ ,$$

with $B_\varepsilon=D^2_{xx}\zeta(x_\varepsilon,y_\varepsilon)=D^2_{yy}\zeta(x_\varepsilon,y_\varepsilon)=-D^2_{xy}\zeta(x_\varepsilon,y_\varepsilon)$. By choosing $r=s$ we obtain that $X\leq Y$. Thus subtracting (C.11) from (C.10) and recalling that $p=q$, we can write the result in the form

$$\text{(C.12)}\qquad 0\leq\delta-1\,.$$

which is a contradiction.

6. If, on the contrary, for all $\varepsilon>0$, $D\zeta(x_\varepsilon,y_\varepsilon)=0$, this means that $x_\varepsilon=y_\varepsilon$ and we have also $D^2\zeta(x_\varepsilon,y_\varepsilon)=0$. But y_ε is a minimum point of $v(\cdot)-u(x_\varepsilon)+\zeta(x_\varepsilon,\cdot)$ and, by applying the definition of viscosity supersolution, we are led to

$$F(-D_y\zeta(x_\varepsilon,y_\varepsilon),-D^2_{yy}\zeta(x_\varepsilon,y_\varepsilon))\geq 1\ ,$$

i.e. $0=F^*(0,0)\geq 1$. This contradiction shows that this case cannot happen either.

The proof of Theorem 4 is now complete. □

C.2 An alternative proof

Here is another, rather different proof of the same result.

1. The function $u^+ := \max(u, 0)$ is still a subsolution of (C.1) as the maximum of two subsolutions (c.f. Perron's method, discussed for example in the User's guide [CIL]). We can therefore assume without loss of generality that $u \geq 0$ on $\overline{\Omega}$ and we will do it in the sequel.

2. We start by extending u and v to $\mathbb{R}^N$ by setting

$$\tilde{u}(x) = \begin{cases} u(x) & \text{in } \overline{\Omega}, \\ 0 & \text{in } (\overline{\Omega})^c, \end{cases}$$

$$\tilde{v}(x) = \begin{cases} v(x) & \text{in } \Omega, \\ 0 & \text{in } \Omega^c. \end{cases}$$

Since u and v are nonnegative on $\overline{\Omega}$, it is clear that $\tilde{u}$ is usc and $\tilde{v}$ is lsc.

We claim that $\tilde{u}$ and $\tilde{v}$ are respectively sub and supersolution of

$$\max(F(Du, D^2u) - 1, u - \psi) = 0 \quad \text{in } \mathbb{R}^N, \tag{C.13}$$

where $\psi(x) = C\mathbb{1}_{\overline{\Omega}}(x)$ with $C > \max(||u||_\infty, ||v||_\infty)$ and F is defined as above.

3. We first show that $\tilde{u}$ is a subsolution of (C.13). This is obvious if $x \in \Omega$ or $x \in (\overline{\Omega})^c$.

Let $x \in \partial\Omega$ be a maximum point of $\tilde{u} - \phi$ for some function $\phi \in C^2(\mathbb{R}^N)$. Two cases may occur : either $\tilde{u}(x) > 0$ but then $\tilde{u}(x) = u(x)$ and x is still a maximum point of $u - \phi$ on $\overline{\Omega}$. Thus since there is loss of boundary condition for the subsolution u at x, we have

$$F_*(D\phi(x), D^2\phi(x)) \leq 1.$$

Moreover the inequality $\tilde{u}(x) \leq \psi^*(x) = C$ holds by construction, thus we get

$$\max(F_*(D\phi(x), D^2\phi(x)) - 1, \tilde{u}(x) - \psi^*(x)) \leq 0\,. \tag{C.14}$$

The other case is when $\tilde{u}(x) = 0$; then it is easy to check that x is also a local maximum point of $0 - \phi$; since 0 is a subsolution of the equation in $\mathbb{R}^N$, this yields

$$F_*(D\phi(x), D^2\phi(x)) \leq 1\,.$$

Thus (C.14) is satisfied.

4. Next we show that $\tilde{v}$ is a supersolution of (C.13). This assertion is almost obvious since (i) on Ω^c, $\psi_* \equiv 0$ and therefore $\tilde{v} \geq \psi_*$ on Ω^c since $v \geq 0$ on $\overline{\Omega}$, and (ii) in Ω the inequality associated to the equation holds. Hence, in both cases, the "max" is nonnegative.

5. The next step consists in comparing $\tilde{u}$ and $\tilde{v}$ in $\mathbb{R}^N$. To do so, we modify $\tilde{u}$ as in the first proof: first we change $\tilde{u}$ to $\tilde{u}_\delta := \delta\tilde{u}$ for $0 < \delta < 1$ close to 1. This function satisfies

$$\max(F(Du, D^2u) - \delta, u - \delta\psi) \leq 0 \quad \text{in } \mathbb{R}^N\,. \tag{C.15}$$

Then, we fix δ (and drop the dependence in δ in $\tilde{u}_\delta$) and, for $\lambda > 1$ close to 1, we introduce the function $\tilde{u}_\lambda(x) := \lambda^{-2}\tilde{u}(\lambda x)$. The function $(\tilde{u}_\lambda)^*$ is a subsolution of

$$\max(F(Du_\lambda, D^2u_\lambda) - \delta, \tilde{u}_\lambda - \psi_{\delta,\lambda}) \leq 0\,. \tag{C.16}$$

where $\psi_{\delta,\lambda}(x) := \delta\lambda^{-2}\psi(\lambda x)$ in $I\!R^N$. The main point is that, because of **(H0)**, we have for λ close enough to 1, $\psi^*_{\delta,\lambda} \le \psi_*$ in $I\!R^N$.

6. Using comparison arguments like those in the first proof and noticing that $\tilde{u}(x) = \tilde{v}(x) = 0$ for $|x|$ large enough (which simplifies matters since we do not have any problem at infinity), it is easy to show that

$$\tilde{u}_\lambda(x) \le \tilde{v}(x) \quad \text{in } I\!R^N . \tag{C.17}$$

Examining the consequences of this inequality for x in $\overline{\Omega}$ yields the result exactly as in the first proof. □

C.3 An example where the minimum exit time is discontinuous

We provide in this section an example where the minimum time function has non-artificial discontinuities at two points of the boundary. This example shows that one cannot have in this case a "strong comparison result" for this problem and that the "weak comparison result" above is optimal. It is worth pointing out anyway that, in this example, the domain Ω is not smooth. It would be interesting to have such a counter-example either for a smooth domain or with an interior discontinuity.

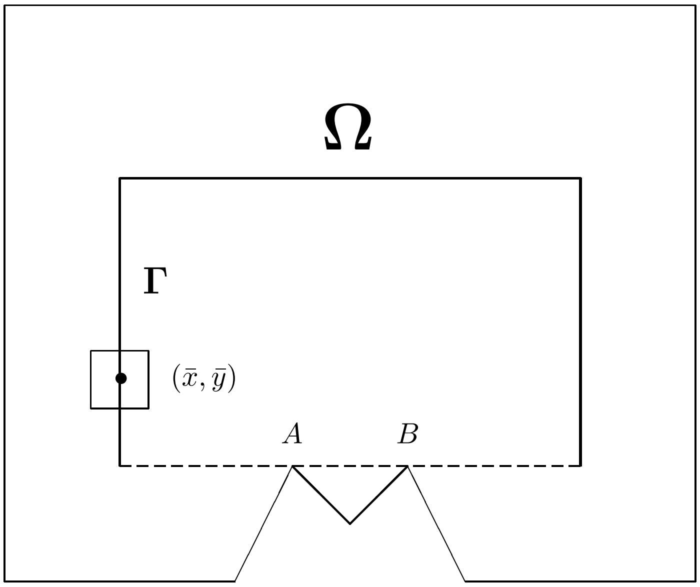

Figure 6: *Paul's limiting exit time is discontinuous at A and B. In fact, it vanishes along the segments joining A to B (and on the whole boundary), but it is bounded away from zero just above the dotted line.*

In order to describe our example, we first introduce the function $\chi : I\!R \to I\!R$ defined by

$$\chi(x) = \begin{cases} x+1 & \text{if } x \le -1 \,, \\ |x|-1 & \text{if } |x| \le 1 \,, \\ -x+1 & \text{if } x \ge 1 \,. \end{cases}$$

Then we introduce the domain $\Omega \subset \mathbb{R}^2$ given, for $R > 0$ large enough, by

$$\Omega := [(-2R, 2R) \times (-R, 2R)] \cap \{(x, y) \in \mathbb{R}^2;\ y > \chi(x)\}$$

(see Figure 6). We claim that the minimum time function has two discontinuities at the points $A := (-1, 0)$ and $B := (1, 0)$. We are going to prove the claim for A, the situation is symmetric for B.

We denote by U the minimum time function. We are going to show that

(i) $\limsup_{\delta \downarrow 0} U(-1, \delta)$ is "large", but

(ii) $\liminf_{\substack{(x,y)\to(-1,0)\\ (x,y)\in\Omega,\ y<0}} U(x, y) = 0$.

We first prove (i) by a Dynamic Programming Principle type argument. To do so, we focus on U^ε, starting the game from any of the points $(-1, \delta)$. Suppose Carol uses the following strategy: if Paul chooses (v_1, v_2), Carol takes $b = 1$ if $v_2 \geq 0$, $b = -1$ if $v_2 < 0$. She persists with this strategy until Paul exits the rectangle $(-R, R) \times (0, R)$. Of course, because of Carol's strategy, Paul can exit this rectangle only on the part of the boundary corresponding to $x = -R$, $x = R$ or $y = R$ (not the part where $y = 0$). We denote by Γ the part of the boundary where Paul can exit.

From the definition of U^ε, we have

$$U^\varepsilon(-1, \delta) \geq U^\varepsilon(\bar{x}, \bar{y}) ,$$

where $(\bar{x}, \bar{y})$ is the first point outside the square $(-R, R) \times (0, R)$ which is achieved by Paul.

If ε is very small, $(\bar{x}, \bar{y})$ is very close to Γ and if R is sufficiently large, the square $(\bar{x}-1, \bar{x}+1) \times (\bar{y}-1, \bar{y}+1)$ is included in Ω and, starting from $(\bar{x}, \bar{y})$, the minimum exit time from Ω is clearly bigger than the minimum exit time from this square; moreover, this minimum exit time from the $(\bar{x}, \bar{y})$–square is independent of $(\bar{x}, \bar{y})$; let's denote it by $\gamma > 0$. It follows that

$$U^\varepsilon(-1, \delta) \geq \gamma .$$

Passing to the limit $\varepsilon \to 0$ we conclude that $U(-1, \delta) \geq \gamma$ for $\delta > 0$. Thus (i) is proved.

To prove (ii), we use the parallel between the minimum time problem and the motion by the signed mean curvature.

First, it is clear that the minimum exit time from Ω is smaller than the minimum exit time from the epigraph of χ, which is infinite everywhere except perhaps on the triangle $\{(x, y);\ y \geq \chi(x) \text{ and } y \leq 0\}$.

To study the exit time of the epigraph of χ, we have to solve the initial value problem

$$w_t - \frac{w_{xx}^+}{1 + w_x^2} = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) ,$$

$$w(x, 0) = \chi(x) \quad \text{in } \mathbb{R} .$$

By standard arguments in viscosity solutions theory, this problem has a unique solution which satisfies

$$\chi(x) \leq w(x, t) \quad \text{in } \mathbb{R} \times (0, +\infty) .$$

Moreover let us denote by $\hat{\chi}$ the concave envelope of χ which is defined in the same way as χ except on the interval $[-1,1]$ where $\hat{\chi} \equiv 0$. Then $\hat{\chi}$ is a supersolution of the problem (this can be proved by easy approximation and regularization arguments) and by classical comparison results

$$w(x,t) \leq \hat{\chi}(x) \quad \text{in } I\!R \times (0,+\infty)\ .$$

Putting all this information together, it is clear that, in order to compute w, it is enough to solve the equation only on the interval $[-1,1]$ with the Dirichlet boundary condition

$$w(1,t) = w(-1,t) = 0 \quad \text{for all } t > 0\ .$$

On this interval where χ is convex, one can even solve the pde with w_{xx} instead of w_{xx}^+ and the Strong Maximum Principle implies that

$$w(x,t) > \chi(x) \quad \text{for any } t > 0\ .$$

But we recall that $\{(x,y);\ y \leq w(x,t)\} = \{(x,y);\ \tilde{U}(x,y) \leq t\}$, where $\tilde{U}$ is the minimum exit time from the epigraph. Therefore the point $(-1,0)$ is the limit of points for which the minimum exit time from the epigraph (and therefore from Ω) is less than t for any t which proves (ii).

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