Vortex Patterns in Ginzburg-Landau Minimizers

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Abstract

We present a survey of results obtained with Etienne Sandier on vortices in the minimizers of the 2D Ginzburg-Landau energy of superconductivity with an applied magnetic field, in the asymptotic regime of large kappa where vortices become point-like. We decribe results which characterize the critical values of the applied field for which vortices appear, their numbers and locations. If the applied field is large enough, it is observed in experiments that vortices are densely packed and form triangular (hexagonal) lattices named Abrikosov lattices. Part of our results is the rigorous derivation of a mean field model describing the optimal density of vortices at leading order in the energy, and then the derivation of a next order limiting energy which governs the positions of the vortices after blow-up at their inter-distance scale. This limiting energy is a logarithmic-type interaction between points in the plane. Among lattice configurations it is uniquely minimized by the hexagonal lattice, thus providing a first justification of the Abrikosov lattice in this regime.

1 Introduction

We are interested in describing mathematical results on vortices in the two-dimensional Ginzburg-Landau model, a celebrated physics model. [1]

Superconductivity consists in the complete loss of resistivity of certain metals and alloys below a certain critical temperature. The consequences of it are the possibility of permanent *superconducting currents* and the particular behavior that, when the material is submitted to an external magnetic field, that field gets expelled from it. A striking phenomenon, predicted by A. Abrikosov, is the possibility of a *mixed state* in type II superconductors where triangular (or hexagonal) vortex lattices appear. Vortices can be described roughly as a quantized amount of vorticity of the superconducting current localized near a point. For reference on the subject, one can refer to standard physics texts, such as [25, 10, 18].

These phenomena have been the object of a huge amount of both experimental and theoretical studies on the physics side. In the last 15 years, the subject has also rapidly developed as a field of interest for mathematicians, who have derived and proved rigorously part of the phenomena observed in the physics, while providing additional information on it (for a survey of that literature, one can see Chapter 14 in [19]).

In addition to its importance in the modelling of superconductivity, the Ginzburg-Landau model turns out to be the simplest case of a gauge theory, and vortices to be the simplest case of topological solitons (for these aspects see [14, 12] and the references therein); moreover, it is mathematically extremely close to the Gross-Pitaevskii model for superfluidity, and models for rotating Bose-Einstein condensates in which quantized vortices are also essential objects, and to which the Ginzburg-Landau techniques have been successfully exported.

Here our aim is to survey some results we have obtained on minimizers of the Ginzburg-Landau model in the "large kappa limit", in the regimes where vortices are expected to appear. For the sake of clarity we will be rather informal in this presentation, however complete proofs can be found in the papers referenced below.

We will provide a description of

- the values of the critical fields for which vortices appear
- the vortex patterns for energy minimizers (ground states)
- the limiting energies which govern their interaction

1.1 The model

Consider a domain Ω in $\mathbb{R}^3$. In the Ginzburg-Landau model, the energy of a superconductor occupying Ω in the presence of a constant applied field H_{ex}, when the exterior region is

[1] It has earned Ginzburg the 2003 Physics Nobel Prize, jointly with Abrikosov for his work on explaining vortex lattices, and Legett for his modelling of superfluidity. Experimental discoveries on superconductivity and Bose-Einstein condensates have also won other physics Nobel prizes.

insulating, is

$$G(u,A) = G_0 + \int_{\mathbb{R}^3} \frac{|\mathrm{curl}\, A - \mathrm{H_{ex}}|^2}{8\pi} + \int_\Omega \frac{1}{2m^*}\left|\left(\hbar\nabla - \frac{ie^*}{c}A\right)u\right|^2 + \alpha|u|^2 + \beta|u|^4. \tag{1.1}$$

In this expression, $u : \Omega \to \mathbb{C}$ is the *order parameter* (generally denoted ψ) whose physical meaning is that of a "wave function" for superconducting electron pairs and $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the electromagnetic vector potential, whose curl is the induced magnetic field. Besides the physical constants $\hbar$ and c, additional constants m^* and e^* are present (see [25] for an explanation of these constants) as well as two quantities α and β that depend on the temperature T and on the superconducting material. Near the so-called critical temperature T_c, it is assumed that β is a positive constant and α is proportional to $T - T_c$ and has the same sign. The quantity G_0 represents the energy of the normal state and does not depend on u or A. From then on, we consider we are below the critical temperature T_c. After some nondimensionalizing procedure (described for example in [19], Chap. 2) and reduction to a two-dimensional domain, the energy functional can be reduced to

$$G_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\mathrm{curl}\ A - h_{ex}|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \tag{1.2}$$

where Ω is now a bounded simply connected domain of $\mathbb{R}^2$.

Here A is now a function from Ω to $\mathbb{R}^2$. The magnetic field in the sample is deduced by $h = \mathrm{curl}\, A = \partial_1 A_2 - \partial_2 A_1$, it is thus a real-valued function in Ω. The notation ∇_A denotes the covariant gradient $\nabla - iA$; $\nabla_A u$ is thus a vector with complex components. u indicates the local state of the material or the phase in the Landau theory approach of phase transitions: $|u|^2$ is the density of "Cooper pairs" of superconducting electrons. With our normalization $|u| \le 1$ and where $|u| \simeq 1$ the material is in the superconducting phase, while where $|u| = 0$, it is in the normal phase (i.e. behaves like a normal conductor), the two phases being able to coexist in the sample.

The *superconducting current* that we will denote j is a real vector given by $j = \langle iu, \nabla_A u\rangle$ where $\langle .,. \rangle$ denotes the scalar-product in $\mathbb{C}$ identified with $\mathbb{R}^2$, it may also be written as

$$\frac{i}{2}\left(u\overline{\nabla_A u} - \bar{u}\nabla_A u\right),$$

where the bar denotes the complex conjugation.

The parameter $h_{\mathrm{ex}} > 0$ represents the intensity of the applied field (assumed to be perpendicular to the plane of Ω). Finally, the parameter ε is the inverse of the "Ginzburg-Landau parameter" usually denoted κ, a non-dimensional parameter depending on the material only. We will be interested in the regime of small ε, corresponding to high-κ (or extreme type-II) superconductors. The limit $\varepsilon \to 0$ or $\kappa \to \infty$ that we will consider is also called the London limit. In this limit, the characteristic size of the vortices, which is ε, tends to 0 and vortices become point-like. In this limit, h_{ex} will be a function of ε and not an independent parameter.

The stationary states of the system are the critical points of G_ε, or the solutions of the Ginzburg-Landau equations :

$$(GL)\begin{cases} -(\nabla_A)^2 u = \dfrac{1}{\varepsilon^2} u(1-|u|^2) & \text{in } \Omega \\ -\nabla^\perp h = \langle iu, \nabla_A u\rangle & \text{in } \Omega \\ h = h_{\text{ex}} & \text{on } \partial\Omega \\ \nu\cdot\nabla_A u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nabla^\perp$ denotes the operator $(-\partial_2, \partial_1)$, and ν the outer unit normal to $\partial\Omega$.

The Ginzburg-Landau equations and functional are invariant under $\mathbb{U}(1)$-gauge-transformations (it is an Abelian gauge-theory):

$$\begin{cases} u \mapsto ue^{i\Phi} \\ A \mapsto A + \nabla\Phi \end{cases} \tag{1.3}$$

The physically relevant quantities are those that are gauge-invariant, such as the energy G_ε, $|u|$, h, etc...

In this paper we will only focus on *global minimizers of the energy* G_ε, in other words ground states. Other results for (stable or unstable) nonminimizing configurations can be found in [19].

1.2 Formal look at the solutions, vortices and critical fields

We start with some very formal considerations on solutions to the (GL) equations (further formal computations can be found in [10] or [19], Chapter 2).

1.2.1 Types of solutions

Three types of solutions (or states) can be found:

1. the normal solution : $(u \equiv 0, \operatorname{curl} A \equiv h_{\text{ex}})$. This is a true solution to (GL) and its energy is very easily computed: it is $\frac{|\Omega|}{4\varepsilon^2}$.

2. the Meissner solution (or superconducting solution) : $(u \equiv 1, A \equiv 0)$. This is a true solution if $h_{\text{ex}} = 0$, and a solution close to this one (i.e. with $|u| \simeq 1$ everywhere) persists if h_{ex} is not too large. Its energy is approximately $G_\varepsilon(1,0) = \frac{h_{\text{ex}}^2}{2}|\Omega|$. By comparing these energies, we see that the Meissner solution is more favorable when h_{ex} is small, while the normal solution is more favorable when h_{ex} is large enough (depending on ε), more precisely when $h_{\text{ex}} > \frac{1}{\varepsilon\sqrt{2}}$.

3. the vortex solutions: there is another state, with vortices, called the *mixed state* where normal and superconducting phases co-exist, and which is more favorable for intermediate values of h_{ex}.

1.2.2 Vortices and their benefit

What are vortices? A vortex is an object centered at an isolated zero of u, around which the phase of u has a nonzero winding number, called the *degree of the vortex.* When ε is small, it is clear from (1.2) that any discrepancy between $|u|$ and 1 is strongly penalized, and a scaling argument hints that $|u|$ is different from 1 only in regions of characteristic size ε. A typical vortex centered at a point x_0 "looks like" $u = \rho e^{i\varphi}$ with $\rho(x_0) = 0$ and $\rho = f(\frac{|x-x_0|}{\varepsilon})$ where $f(0) = 0$ and f tends to 1 as $r \to +\infty$, i.e. its characteristic core size is ε, and

$$\frac{1}{2\pi}\int_{\partial B(x_0,R\varepsilon)} \frac{\partial\varphi}{\partial\tau} = d \in \mathbb{Z}$$

is an integer, called the *degree of the vortex.* For example $\varphi = d\theta$ where θ is the polar angle centered at x_0 yields a vortex of degree d.

True radial solutions in $\mathbb{R}^2$ of the Ginzburg-Landau equations of degree n, of the form

$$u_n(r,\theta) = f_n(r)e^{in\theta}, \quad A_n(r,\theta) = g_n(r)(-\sin\theta, \cos\theta)$$

have been shown to exist [15, 16, 5]. In a bounded domain, there are solutions with several such vortices glued together, for example arranged along a triangular lattice (their existence is proved at least as a bifurcation from the normal solution, see [8, 3]).

For a configuration with vortices we have the important (formal) relation

$$\operatorname{curl}\nabla\varphi = 2\pi\sum_i d_i\delta_{a_i} \tag{1.4}$$

where the a_i's are the centers of the vortices and the d_i's their degrees. This hints at the reason why vortices allow to gain energy. Indeed the first term in the energy $\int|\nabla_A u|^2$ can be written $\int|u|^2|\nabla\varphi - A|^2$ so this term prefers $\nabla\varphi$ to be close to A, while the second term in the energy prefers h to be close to h_{ex}. But since $\operatorname{curl} A = h$ this means that $\operatorname{curl}\nabla\varphi$ should be close to h_{ex}. With the above relation (1.4), this translates into

$$2\pi\sum_i d_i\delta_{a_i} \text{ is as close as possible to the constant value } h_{\mathrm{ex}}. \tag{1.5}$$

One sees that if there are no vortices the sum on the left-hand side is zero hence this will fail to be achieved if h_{ex} is large enough; on the contrary one sees how vortices can "help" energetically. Then, the question of understanding what (1.5) may exactly mean and in what sense, and what configurations of points a_i satisfy this assertion, is the core of the matter of our study. We can say right away that the conjecture is that (just like in the optimal packing problem) the configuration of vortex points a_i (with degrees $d_i = 1$) which optimizes (1.5), i.e. which best approximates the constant value h_{ex}, is the hexagonal Abrikosov lattice of density h_{ex}. However, as we shall see below, the situation is a bit more subtle than that, since there are some boundary effects which we have neglected in this formal discussion.

1.2.3 Critical fields

Let us now be more precise about the situation as a function of h_{ex}, as is known from the physics literature and the mathematical studies.

There are three main critical values of h_{ex} or *critical fields* H_{c_1}, H_{c_2}, and H_{c_3}, for which phase-transitions occur.

- For $h_{\text{ex}} < H_{c_1}$ the energy minimizer is the superconducting (Meissner) solution. H_{c_1} is of order of $|\log \varepsilon|$ as $\varepsilon \to 0$.
- For $h_{\text{ex}} = H_{c_1}$ the first vortice(s) appear.
- For $H_{c_1} < h_{\text{ex}} < H_{c_2}$ the superconductor is in the mixed phase i.e. there are vortices. The higher $h_{\text{ex}} > H_{c_1}$, the more vortices there are. The vortices repel each other so they tend to arrange in these hexagonal Abrikosov lattices in order to minimize their repulsion.
- For $h_{\text{ex}} = H_{c_2} \sim \frac{1}{\varepsilon^2}$, the vortices are so densely packed that they overlap each other, and a second phase transition occurs, after which $|u| \sim 0$ inside the sample, i.e. all superconductivity in the bulk of the sample is lost.
- For $H_{c_2} < h_{\text{ex}} < H_{c_3}$ superconductivity persists only near the boundary, this is called *surface superconductivity.*
- For $h_{\text{ex}} > H_{c_3} = O(\frac{1}{\varepsilon^2})$ (defined in decreasing fields), the sample is completely in the normal phase $u \equiv 0$.

In Section 2 we give a precise mathematical description of that picture for all h_{ex} much smaller than H_{c_2}, at leading order of the energy. In Section 3 we present more recent results where we refine this study in the cases with a large number of vortices, to derive a next order interaction energy which governs the vortex patterns.

2 Main results for global minimization at the leading order

In all that follows the notation $a = O(b)$ will mean that a/b remains bounded, and $a \ll b$ that $a/b \to 0$, in the limit $\varepsilon \to 0$.

2.1 The vorticity measures

Recall that a complex-valued map u can be written in polar coordinates $u = \rho e^{i\varphi}$ with a phase φ which can be multi-valued. Given a configuration (u, A), we define its *vorticity* by

$$\mu(u, A) = \operatorname{curl} \langle iu, \nabla_A u \rangle + \operatorname{curl} A = \operatorname{curl} j + h, \tag{2.1}$$

where we recall that $j = \langle iu, \nabla_A u\rangle$ is the superconducting current. Formally, considering that $\rho = |u| \simeq 1$ we have $\mu(u,A) = \operatorname{curl}(\rho^2(\nabla\varphi - A)) + \operatorname{curl} A \simeq \operatorname{curl}\nabla\varphi$, so using (1.4), we have the approximate (formal) relation

$$\mu(u,A) = 2\pi \sum_i d_i \delta_{a_i} \tag{2.2}$$

where a_i's are the vortices of u and d_i's their degrees, and δ the Dirac mass. This relation can be justified rigorously in some appropriate sense, see [13] or [19], Theorem 6.1. Thus we see why the gauge-invariant quantity $\mu(u,A)$ is appropriate as a proxy for the vortices of u (it is formally like the vorticity for fluids).

Remark 2.1. *When the second of the Ginzburg-Landau equations* (GL) *is satisfied, taking its curl, we find that the vorticity and the induced field are linked by the London equation (this time an identity and not an approximation)*

$$\begin{cases} -\Delta h + h = \mu(u,A) & \text{in } \Omega \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Thus the knowledge of the vorticity is equivalent to that of the induced field h.

2.2 Global minimizers of G_ε close to H_{c_1}

Let us introduce h_0 the solution of

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega \\ h_0 = 1 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

and

$$\lambda_\Omega = (2\max|h_0 - 1|)^{-1}. \tag{2.5}$$

We also introduce the set $\Lambda = \{x \in \Omega / h_0(x) = \min h_0\}$ and we will assume here for simplicity that it is reduced to only one point called $\bar p$, and denote $Q(x) = \langle D^2 h_0(\bar p)x, x\rangle$ its second order differential, assumed to be definite positive.

The first vortices will appear near the point $\bar p$, and in order to describe them, we will perform blow-ups around $\bar p$ at suitable scales.

With these notation, a first essential result is the asymptotic formula for H_{c_1} (confirming physical predictions that H_{c_1} is of the order of $|\log\varepsilon|$ as $\varepsilon \to 0$):

$$H_{c_1} = \lambda_\Omega |\log\varepsilon| + cst. \tag{2.6}$$

Theorem 1 ([23, 19])**.** *There exists an increasing sequence of values*

$$H_n = \lambda_\Omega |\log\varepsilon| + (n-1)\lambda_\Omega \log\frac{|\log\varepsilon|}{n} + \textit{constant order terms}$$

such that if $h_{\rm ex} \le H_{c_1} + O(\log\ |\log\varepsilon|)$ *and* $h_{\rm ex} \in (H_n, H_{n+1})$*, then global minimizers of* G_ε *have exactly* n *vortices of degree 1, at points* $a_i^\varepsilon \to \bar{p}$ *as* $\varepsilon \to 0$*, and the* $\tilde{a_i^\varepsilon} = \sqrt{\frac{h_{\rm ex}}{n}}(a_i^\varepsilon - \bar{p})$ *converge as* $\varepsilon \to 0$ *to a minimizer of*

$$w_n(x_1, \cdots, x_n) = -\pi \sum_{i\neq j} \log|x_i - x_j| + \pi n \sum_{i=1}^{n} Q(x_i). \tag{2.7}$$

Through this theorem we see that the behavior is as expected: below $H_{c_1} = H_1$ there are no vortices in energy minimizers, then at H_{c_1} the first vortex becomes favorable, close to the point $\bar{p}$. Then, there is a sequence of additional critical fields $H_2, H_3, ...$ separated by increments of $\log\ |\log\varepsilon|$, for which a second, third, ..., vortex becomes favorable. Each time the optimal vortices are located close to $\bar{p}$ as $\varepsilon \to 0$ (cf. Fig. 1) and after

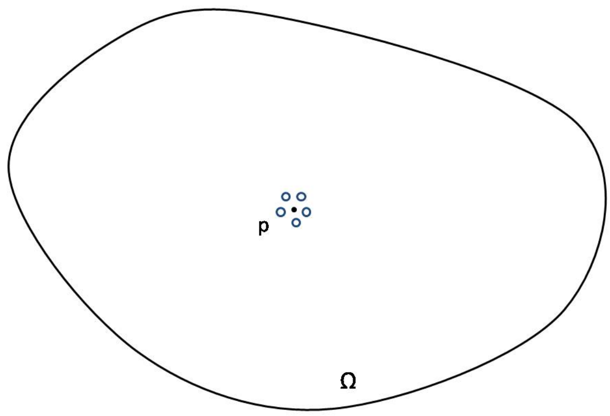

Figure 1: Minimizers with small number of vortices

blowing-up at the scale $\sqrt{\frac{h_{\rm ex}}{n}}$ around $\bar{p}$, they converge to configurations which minimize w_n in $\mathbb{R}^2$. Now, w_n, which appears as a limiting energy (after that rescaling) contains a repulsion and a confinement term. It is a standard two-dimensional interaction, however rigorous results on its minimization are hard to obtain as soon as $n \ge 3$. When Q has rotational symmetry, numerical minimization (see Gueron-Shafrir [11]) yields very regular shapes (regular polygons for $n \le 6$, regular stars) which look very much like the birth of a triangular lattice as n becomes large (their density tends to be uniform supported in a fixed disc of $\mathbb{R}^n$ as $n \to \infty$), see Fig. 2.

All these results are in very good agreement with experimental observations.

Remark 2.2. *It was proved in [24] that for* $h_{\rm ex} < H_{c_1}$*, the energy-minimizer is unique and has no vortex.*

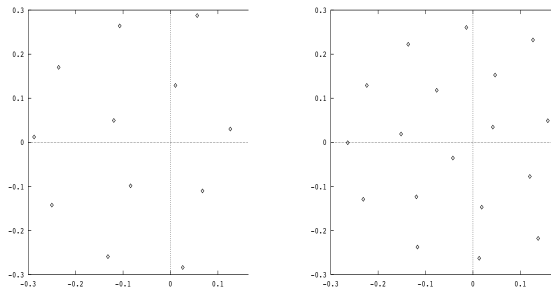

Figure 2: Results of the numerical optimization of [11] for w_n, $n = 16$ and $n = 21$.

2.3 Global minimizers in the intermediate regime

In the next higher regime of applied field, the result is the following:

Theorem 2 ([19]). *Assume $h_{\rm ex}$ satisfies as $\varepsilon \to 0$,*

$$\log\ |\log \varepsilon| \ll h_{\rm ex} - H_{c_1} \ll |\log \varepsilon|$$

then there exists $1 \ll n_\varepsilon \ll h_{\rm ex}$ such that

$$h_{\rm ex} \sim \lambda_\Omega \left(|\log \varepsilon| + n_\varepsilon \log \frac{|\log \varepsilon|}{n_\varepsilon} \right)$$

and if $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε, then

$$\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n_\varepsilon} \longrightarrow \mu_0 \quad \textit{in the weak sense of measures}$$

where $\tilde{\mu}(u_\varepsilon, A_\varepsilon)$ is the push-forward of the vorticity measure $\mu(u_\varepsilon, A_\varepsilon)$ under the blow-up $x \mapsto \sqrt{\frac{h_{\rm ex}}{n_\varepsilon}}(x - \bar{p})$, and μ_0 is the unique minimizer over probability measures of

$$I(\mu) = -\pi \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x - y|\, d\mu(x)\, d\mu(y) + \pi \int_{\mathbb{R}^2} Q(x)\, d\mu(x). \tag{2.8}$$

Here, n_ε corresponds to the expected optimal number of vortices. Note that from (2.2) we have

$$\tilde{\mu}(u_\varepsilon, A_\varepsilon) \simeq 2\pi \sum_i d_i \delta_{\tilde{a}_i}$$

where the $\tilde{a}_i$'s are the images of the true vortices of $(u_\varepsilon, A_\varepsilon)$ after the blow-up.

The problem of minimizing I is a classical one in potential theory. Its minimizer μ_0 is a probability measure of constant density over a subdomain of $\mathbb{R}^2$ (typically a disc or an ellipse). This result is in continuous connection with Theorem 1, except $n_\varepsilon \gg 1$. Again, vortices in the minimizers converge to $\bar{p}$ as $\varepsilon \to 0$, and when one blows up at the right scale $\sqrt{\frac{h_{\mathrm{ex}}}{n_\varepsilon}}$ around $\bar{p}$, one obtains a uniform density of vortices in a subdomain of $\mathbb{R}^2$ (a disc if $D^2 h_0$ has rotational symmetry).

2.4 Global minimizers in the regime n_ε proportional to h_{ex}

This happens in the next regime: $h_{\mathrm{ex}} \sim \lambda|\log \varepsilon|$ with $\lambda > \lambda_\Omega$. Let us define

$$E_\lambda(\mu) = \frac{1}{2\lambda}\int_\Omega |\mu| + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2,$$

over bounded Radon measures such that $E_\lambda(\mu) < \infty$, where $\int_\Omega |\mu|$ is the total mass of the measure μ and

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

is the magnetic field associated to μ as in (2.3).

In this regime, the region filled up with vortices is no longer concentrating just around the point $\bar{p}$ but is spread out at finite distance from $\bar{p}$, so there is no need to blow-up in order to distinguish the vortices. However, we still need to divide or normalize the vorticity $\mu(u, A)$ by the order of the expected number of vortices, which is h_{ex}, and blows up with $\varepsilon \to 0$.

Theorem 3 ([21, 19]). *Assume $h_{\mathrm{ex}} = \lambda|\log \varepsilon|$ where $\lambda > 0$ is a constant independent of ε. If $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε, then as $\varepsilon \to 0$*

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}} \to \mu_* \quad \textit{in the weak sense of measures}$$

where μ_* is the unique minimizer of E_λ.

Observe also that E_λ can be rewritten

$$E_\lambda(\mu) = \frac{1}{2\lambda}\int_\Omega |\mu| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)\, d(\mu-1)(x)\, d(\mu-1)(y) \tag{2.10}$$

where G is a Green's function, the solution to $-\Delta G + G = \delta_y$ with $G = 0$ on $\partial\Omega$. That way, the similarity with I is more apparent.

There remains to minimize E_λ. It turns out that this problem is dual in the sense of convex duality to an obstacle problem:

Proposition 2.1. *μ minimizes E_λ if and only if h_μ is the minimizer for the variational problem*

$$\min_{\substack{h \geq 1 - \frac{1}{2\lambda} \\ h=1 \text{ on } \partial\Omega}} \int_\Omega |\nabla h|^2 + h^2. \tag{2.11}$$

Now, the solution of the obstacle problem (2.11) is well-known, and given by a variational inequality. Obstacle problems are a particular type of *free-boundary problems*, the free-boundary here being the boundary of the so-called "coincidence set"

$$\omega_\lambda = \left\{ x \in \Omega / h_{\mu_*}(x) = 1 - \frac{1}{2\lambda} \right\} = \text{supp } \mu_*. \tag{2.12}$$

Then h_{μ_*} verifies $-\Delta h_{\mu_*} + h_{\mu_*} = 0$ outside of ω_λ, so ω_λ is really the support of μ_*, on which μ_* is equal to the constant density $1 - \frac{1}{2\lambda}$ so we may write $\mu_* = (1 - \frac{1}{2\lambda})\mathbf{1}_{\omega_\lambda}$, see Fig. 3. μ_* is thus completely characterized (ω_λ is itself a set uniquely determined by λ via (2.11)).

An easy analysis of this obstacle problem yields the following:

1. $\omega_\lambda = \varnothing$ (hence $\mu_* = 0$) if and only if $\lambda < \lambda_\Omega$, where λ_Ω was given by (2.5). (This corresponds to the case $h_{\text{ex}} < H_{c_1}$.)

2. For $\lambda = \lambda_\Omega$ then $\omega_\lambda = \{\bar{p}\}$. This is the case when $h_{\text{ex}} \sim H_{c_1}$ at leading order. In the scaling chosen here $\mu_* = 0$ but the true behavior of the vorticity is ambiguous unless going to the next order term as done in Theorems 1 and 2.

3. For $\lambda > \lambda_\Omega$, the measure of ω_λ is nonzero, so the limiting vortex density $\mu_* \neq 0$. Moreover, as λ increases (i.e. as h_{ex} does), the set ω_λ increases. When $\lambda = +\infty$, ω_λ becomes Ω and $\mu_* = 1$, this corresponds to the case $h_{\text{ex}} \gg |\log \varepsilon|$ of the next subsection.

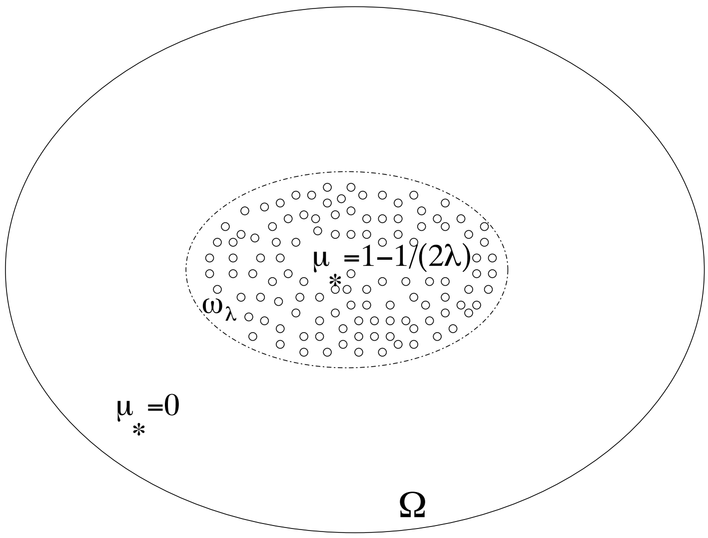

Figure 3: Optimal density of vortices according to the obstacle problem.

2.5 Global minimizers in the regime $|\log\,\varepsilon| \ll h_{\rm ex} \ll \varepsilon^{-2}$

For applied fields larger than $\frac{1}{\varepsilon^2}$ but below H_{c_2}, the parameter $\lambda = \frac{h_{\rm ex}}{|\log\,\varepsilon|}$ above is formally $+\infty$ and we find $\min E_\infty = 0$ and $\mu_* = 1$. The vortex density is thus found to be uniform but the expansion $\min G \sim h_{\rm ex}^2 E_\infty(\mu_*)$ is degenerate. However, even though the number of vortices becomes very large, the minimization problem becomes local and can be solved by blowing-up and using Theorem 3. The minimal energy density can then be evaluated:

Theorem 4 ([20, 19])**.** *Assume, as $\varepsilon \to 0$, that $|\log\,\varepsilon| \ll h_{\rm ex} \ll 1/\varepsilon^2$. Then, letting $(u_\varepsilon, A_\varepsilon)$ minimize G_ε, and letting $g_\varepsilon(u, A)$ denote the energy-density $\frac{1}{2}\left(|\nabla_A u|^2 + |h - h_{\rm ex}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)$, we have*

$$\frac{2g_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\rm ex}\log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}} \to dx \quad \text{as } \varepsilon \to 0$$

in the weak sense of measures, where dx denotes the two-dimensional Lebesgue measure; and thus

$$\min G_\varepsilon(u, A) \sim \frac{|\Omega|}{2} h_{\rm ex} \log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \quad \text{as } \varepsilon \to 0,$$

where $|\Omega|$ is the area of Ω. Moreover

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\rm ex}} \to dx \quad \text{in the weak sense of measures.}$$

2.6 Methods of the proofs

The method we use to obtain those results is the scheme of "Γ-convergence", consisting in evalutating very precisely the minimal energy via upper bounds (obtained by an explicit construction and computation) and matching lower bounds which are *ansatz-free*. This requires being able to describe very precisely the energy carried by vortices even if there is a very large number of them. The method for that originates in the book [6] where situations with bounded numbers of vortices were studied in a simplified context (without magnetic field). Later on, a more generally applicable "technology" was developed by Jerrard and Sandier independently to obtain estimates of self-interaction energy of vortices even when their number if unbounded. The main estimate is that each vortex of degree d carries an energy at least $\pi|d|\log\frac{r}{\varepsilon}$ where r is the distance to its nearest neighbors (for more precise estimates see Theorem 4.1 in [19]). The rest of our analysis then consists in comparing via appropriate splittings of the Ginzburg-Landau functional, this self-interaction energy cost and the vortex-replusion cost to the energy "benefit" of the vortices due to their interaction with the magnetic field.

2.7 Summary

In the results above we have identified the critical fields and regimes for which vortices appear, and have characterized the optimal vortex densities *at leading order*, i.e. derived either limiting interaction energies or mean field models in the cases of Theorems 3 and 4. More precisely we have identified the following regimes:

1. When $h_{\rm ex} < H_{c_1}$ there are no vortices.

2. At $h_{\rm ex} = H_{c_1} = H_1$ one vortex of degree 1 appears, near the point $\bar{p}$. As $h_{\rm ex}$ crosses $H_2, H_3, \cdots, H_n$ an n-th vortex of degree 1 appears, also near $\bar{p}$. After blow-up around $\bar{p}$ these n vortices tend to arrange according to regular shapes minimizing w_n (see again Fig. 1, 2).

3. When $\log\ |\log \varepsilon| \ll h_{\rm ex} - H_{c_1} \ll |\log \varepsilon|$ then the number of vortices n is no longer bounded with respect to ε, but remains $\ll h_{\rm ex}$. There is then a "cloud" of vortices around the point $\bar{p}$, and when blown-up at the scale $\sqrt{\frac{h_{\rm ex}}{n}}$ this cloud appears as a uniform density supported in an ellipse.

4. When $h_{\rm ex} = \lambda |\log \varepsilon|$ with $\lambda > \lambda_\Omega$ there is a cloud of vortices with uniform density (proportional to $h_{\rm ex}$) over the subdomain ω_λ (completely determined by λ), and essentially no vortices outside ω_λ, as in Fig. 3.

5. When $|\log \varepsilon| \ll h_{\rm ex} \ll \frac{1}{\varepsilon^2}$ there is a uniform cloud of vortices covering up the whole domain Ω with a constant density $h_{\rm ex}$.

3 Next-order behavior of vortices

Going back to the conclusions above, the behavior of individual vortices is completely understood in the regimes corresponding to items 1,2 where the number of vortices is finite and fixed. In items 3,4,5 the number of vortices blows up as $\varepsilon \to 0$, and we only know their optimal average density. Such a constant optimal density is in agreement with the Abrikosov lattice where the average density of vortices is also constant, however it is far from justifying the presence of a lattice. Many other patterns (starting with non-hexagonal lattices) are admissible. In other words the mean field description above is insensitive to the precise pattern formed by vortices. This pattern is in fact, as we shall see, selected by the minimization of the next term in the asymptotic expansion of the energy as $\varepsilon \to 0$. This is the object of what follows. Proving this requires more sophisticated tools to obtain more precise estimates on the energy of vortices than the $\pi |d| \log \frac{r}{\varepsilon}$ equivalent mentioned above (in fact we really need to estimate this vortex energy up to a $o(1)$ error). In order to understand the vortex patterns, which are really driven by the next order interaction term, one then needs to blow up (i.e. zoom in) the solutions in space at the scale where one sees individual well-separated vortices.

3.1 Splitting of the energy and blow-up

The next order expansion of the energy is achieved by finding a splitting of the energy which separates, via an identity, the leading order term found above from a remainder.

This splitting states (roughly) that

$$G_\varepsilon(u,A) = h_{\rm ex}^2 E_\lambda(\mu_*) + G_1(u,A) \tag{3.1}$$

where $G_1(u,A)$ is a new functional but of order $h_{\rm ex}$, hence a next order correction, since $h_{\rm ex} \to +\infty$ as $\varepsilon \to 0$ in the regime of interest. In the case $h_{\rm ex} \gg |\log \varepsilon|$ this should be replaced by

$$G_\varepsilon(u,A) = \frac{|\Omega|}{2} h_{\rm ex} \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}} + G_1(u,A) \tag{3.2}$$

where again $G_1(u,A)$ is of order $h_{\rm ex}$. Minimizers of G are obviously the same as minimizers of G_1 hence there remains to minimize G_1.

The results we obtain below are valid for all the regimes 3,4,5 above. However, for the sake of simplicity of the presentation, we will sometimes reduce to case 5, where we have a uniform density of vortices in Ω, and to the simpler periodic setting.

We explain the proof of the energy splitting (3.1)–(3.2) in the periodic situation where it is much simpler: assuming we are in a torus and all the gauge-invariant quantities are periodic, let $h = \operatorname{curl} A$ and set $h_1 = h - h_{\rm ex} + \frac{1}{2}\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}$. Inserting into (1.2) and expanding the square terms we find

$$G_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + h_1^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + \frac{1}{8}\left(\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\right)^2 |\Omega| - \frac{1}{2}\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\int_\Omega h_1.$$

On the other hand we have the London equation (2.3) $-\Delta h + h = \mu(u,A)$ so

$$-\Delta h_1 + h_1 = \mu(u,A) - h_{\rm ex} + \frac{1}{2}\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \tag{3.3}$$

and since we are in a periodic situation it follows that

$$\int_\Omega h_1 = (-h_{\rm ex} + \frac{1}{2}\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}})|\Omega| + \int_\Omega \mu(u,A).$$

Inserting into the above we find

$$G_\varepsilon(u,A) = \frac{|\Omega|}{2} h_{\rm ex}\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}} - \frac{1}{8}\left(\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\right)^2 |\Omega| + G_1(u,A) \tag{3.4}$$

where

$$G_1(u,A) = \left[\frac{1}{2}\int_\Omega |\nabla_A u|^2 + h_1^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right] - \frac{1}{2}\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\int_\Omega \mu(u,A). \tag{3.5}$$

This is actually also the functional G_1 that we find in the non-periodic case. Note that the constant leading order term in (3.4) is the same as that found in Theorem 4.

Let us continue to restrict for simplicity to this periodic case with $h_{\mathrm{ex}} \gg |\log \varepsilon|$. Since the optimal vortex density is h_{ex}, the typical intervortex distance is $\frac{1}{\sqrt{h_{\mathrm{ex}}}}$ so we should blow up the configurations by $\sqrt{h_{\mathrm{ex}}}$ to see well-separated vortices in the plane, with an average density equal to 1. Then, let us take a point x at random in the domain, and blow-up at the scale $\sqrt{h_{\mathrm{ex}}}$ around this point. More precisely set $H(y) = h_1(x + \frac{y}{\sqrt{h_{\mathrm{ex}}}})$. Scaling appropriately in the equation (3.3) we find (using that $\log \frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}} \ll h_{\mathrm{ex}}$) that the limit of H as $\varepsilon \to 0$ should satisfy

$$-\Delta H = 2\pi \sum_i d_i \delta_{p_i} - 1 \tag{3.6}$$

where this time the p_i's are the images of the true vortices a_i of u under the blow up around x at scale $\sqrt{h_{\mathrm{ex}}}$. These should now be well-separated points in the whole plane $\mathbb{R}^2$, and the relation (3.6) holds in all $\mathbb{R}^2$, the sum being infinite. The complication is that we have one such relation for each choice of blow-up center x.

3.2 The "renormalized" energy

The next question is then to understand how the energy G_1 governs the interaction between these limiting blown-up vortex points p_i. One can notice that (3.5) is the difference between two terms: a first positive energy which looks very much like a Ginzburg-Landau energy with zero external magnetic field, and a second term which from (2.2) is roughly equal to $-\pi \log \frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}} \sum_i d_i$ where the d_i's are the degrees of the vortices. Moreover as mentioned above it is expected (see [19]) that in the positive Ginzburg-Landau part, each vortex at distance $\frac{1}{\sqrt{h_{\mathrm{ex}}}}$ from its nearest neighbours has a cost (or self-interaction energy) $\sim \pi d_i^2 \log \frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}$. So this cost gets cancelled by the other term, provided $d_i = 1$. Thus if we do not have $d_i = 1$ the total cost of one vortex blows up, so energetically only vortices of degrees $+1$ are favorable. So without loss of generality we may assume that all the d_i's above are $+1$. Then the cost of each vortex should be exactly compensated by the $-\pi \log \frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}} \sum_i d_i$ term. If this can all be made rigorous, what should be left after this substraction of two "infinite" costs should be an energy of lower order, corresponding just to the interaction cost between the vortices, or a "renormalized energy" cost (we use this expression by analogy with [6]). Extracting this energy thus requires very precise energy estimates. The second thing one can notice is that from the second (GL) equation we have $j = -\nabla^\perp h = -\nabla^\perp h_1$ with $j = |u|^2(\nabla\varphi - A)$ and so $\int_\Omega |\nabla_A u|^2$ is approximately equal to $\int_\Omega |\nabla h|^2 = \int_\Omega |\nabla h_1|^2$. So the energy left in G_1 should be expressable in terms of $\int_\Omega |\nabla H|^2$. It is in fact more convenient below to express things in terms of j.

We are now in a position to present the renormalized energy that we derive through this procedure.

Definition 1. *Let j be a vector field in $\mathbb{R}^2$ and ν a positive measure on $\mathbb{R}^2$. We say (j, ν) belongs to the admissible class $\mathcal{A}$ if*

$$\nu = 2\pi \sum_{p \in \Lambda} \delta_p \quad \text{for some discrete set } \Lambda \subset \mathbb{R}^2,$$

$$\lim_{R\to\infty} \frac{\nu(B_R)}{|B_R|} = 1 \tag{3.7}$$

and

$$\operatorname{curl} j = \nu - 1 = 2\pi \sum_{p\in\Lambda} \delta_p - 1 \qquad \operatorname{div} j = 0. \tag{3.8}$$

We denote by χ_{B_R} cutoff functions associated to the family of balls B_R centered at the origin and of radius R such that

$$|\nabla \chi_{B_R}| \le 2 \quad \chi_{B_R}(x) \equiv 1 \text{ in } B_{R-1} \quad \chi_{B_R}(x) = 0 \text{ outside } B_R. \tag{3.9}$$

Definition 2. *The renormalized energy W is defined, for $(j,\nu) \in \mathcal{A}$, by*

$$W(j) = \limsup_{R\to\infty} \frac{W(j,\chi_{B_R})}{|B_R|}, \tag{3.10}$$

(we denote the dependence as of j only since ν can be deduced from j via (3.8)*) where*

$$W(j,\chi_{B_R}) = \lim_{\eta\to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi_{B_R} |j|^2 + \pi \log \eta \sum_{p\in\Lambda} \chi_{B_R}(p) \right). \tag{3.11}$$

Observe that the limiting j (limit of the superconducting current j_ε after blow-up and $\varepsilon \to 0$) should be equal to $-\nabla^\perp H$ where H satisfies (3.6). But solutions to (3.6) in dimension 2 have a logarithmic singularity at each point p_i, thus $\int |\nabla H|^2 = \int |j|^2$ diverges near each vortex point p. This is why in (3.11) small holes need to be cut out around each p. Adding the expected logarithmic divergence $\pi \log \eta$ for each vortex and letting η tend to 0 is like "renormalizing" and taking the regular part of a Green's function, hence the need for the parameter η. The cut-off χ_R is just there to avoid boundary effects, since otherwise $W(j, \mathbf{1}_{B_R})$ would oscillate wildly between $+\infty$ and $-\infty$ as a point p tends to ∂B_R.

In the end W is a logarithmic interaction between points in the plane. It behaves like $-\pi \log |p_i - p_j|$ when $|p_i - p_j| \to 0$. However it is not only an interaction between the points p's but rather the interaction between these points acting like charged particles and between them and the fixed background constant "charge" -1 (see again (3.6)), averaged over larger and larger balls, see Fig. 4.

We will comment more on the minimization of W after the statement of our last main theorem.

3.3 Main result

For the statement of the theorem we return to a bounded domain and general values of $h_{\rm ex}$. The blow-up procedure should now be made around any point of ω_λ, the support of the limiting vortex density, defined in (2.12). Moreover, since the expected limiting vortex density is no longer $h_{\rm ex}$ but $(1-\frac{1}{2\lambda})h_{\rm ex}$, the blow-up should be made at a slightly modified scale.

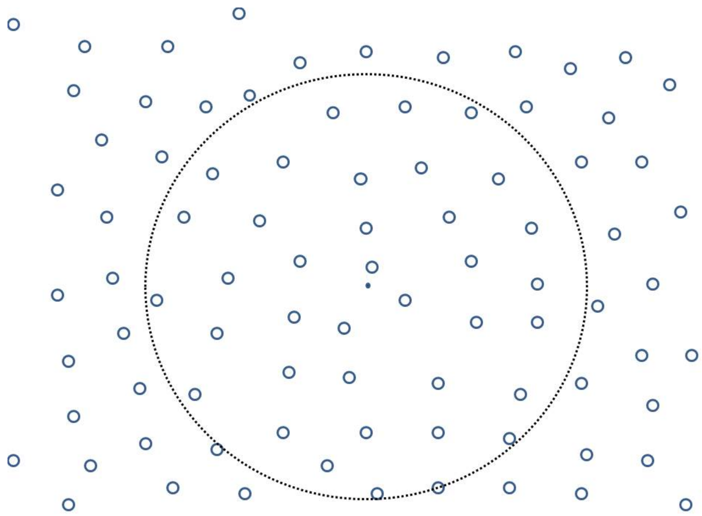

Figure 4: The energy of a collection of blown-up vortices in the whole plane (with density 1 at infinity) is calculated by averaging over larger and larger balls

Theorem 5 ([22])**.** *Assume that*

$$\lambda \in (\lambda_\Omega, +\infty] \text{ and } h_{\rm ex} = \lambda |\log \varepsilon|, \tag{3.12}$$

and if $\lambda = +\infty$ *that* $h_{\rm ex} \ll \frac{1}{\varepsilon^2}$. *Let* $(u_\varepsilon, A_\varepsilon)$ *be a minimizer of* G_ε *and for* $x \in \omega_\lambda$ *let*

$$\tilde{j}_{\varepsilon,x}(\cdot) = \frac{1}{\sqrt{(1-1/(2\lambda))h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)\left(x + \frac{\cdot}{\sqrt{(1-1/(2\lambda)h_{\rm ex}}}\right)$$

be the blow-up image of the superconducting current at the appropriate scale. There exists a probability measure P *on vector fields on* $\mathbb{R}^2$ *such that the following hold:*

1. *up to extraction, for any continuous function* Φ *on the space of vector fields on* $\mathbb{R}^2$*, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{|\omega_\lambda|} \int_{\omega_\lambda} \Phi(\tilde{j}_{\varepsilon,x})\, dx = \int \Phi(j)\, dP(j) \tag{3.13}$$

2. P*-almost every* j *satisfies*

$$\mathrm{curl}\, j = 2\pi \sum_{p\in\Lambda} \delta_p - 1, \qquad \mathrm{div}\ j = 0,$$

for some discrete subset Λ *of* $\mathbb{R}^2$

3. P*-almost every* j *minimizes the function* W *of* (3.10).

Moreover, we have

$$\min G_1 = (1 - \frac{1}{2\lambda})h_{\rm ex}|\omega_\lambda| \min W + o(h_{\rm ex})$$

so $\min G$ *may be computed up to* $o(h_{\rm ex})$.

The above theorem may seem a bit abstract however its concrete meaning is the following : *if one considers an energy minimizer and picks a blow up point at random and looks at the blown-up profile of vortices around that point, then almost surely, one sees a minimizer of* W.

The need for probability measures is due to the difficulty in localizing energy lower bounds since after blown-up one is on an unbounded domain. The probability measure approach allows to do this via the use of the ergodic theorem. Note that from the characterization (3.13) this probability measure is like a Young measure, but in contrast with Young measures, it is not a probability measure on values taken by the functions, but rather a probability measure on the whole limiting profile around a point, so it contains more information. Our approach can be seen as an alternate to the related approach in [2].

Looking again at (3.6) and viewing W as a "renormalized" computation of $\int |\nabla H|^2$, we see that W measures in some sense the size of $2\pi \sum_i \delta_{p_i} - 1$. So we are back (but at the blown-up scale) to the question of (1.5): W is a measure of how close this sum of Diracs is to a constant, and W should be minimized. The open question is then to know whether, like in packing problems, W is minimized by configurations of points which form a triangular lattice. This is a question of crystallisation, for a logarithmic type of interaction, and as such, it is known to be very difficult to answer. Even proving that minimizers have some periodicity seems out of reach.

However, we can answer an easier question: what are minimizers of W if one restricts to pure lattice configurations?

Theorem 6 ([22]). *The minimimum of* W *over lattices of volume* 2π *is uniquely achieved by the hexagonal (or triangular) lattice.*

A simple proof can be found using results from number theory. Indeed let H be a solution to $-\Delta H = \delta_0 - 1$ on a torus of volume 1 of arbitrary shape. The expression (3.11) simplies quite a bit under this periodicity assumption and we can Fourier transform the explicit expression for W in that case to make it a function of the lattice. It then becomes a regularisation of the divergent series $\sum_{p\in\Lambda} \frac{1}{|p|^2}$. By some transformations on modular functions, minimizing W becomes equivalent to minimizing the Epstein zeta function $\zeta(s) = \sum_{p\in\Lambda} \frac{1}{|p|^s}$ with $s > 2$, over lattices Λ with fixed volume. Results from number theory from the 60's [7, 17] say that this is uniquely minimized by the hexagonal lattice.

So we conclude that at least W allows to distinguish between lattices. We have thus derived a nontrivial variational problem for the emergence of Abrikosov lattices, and provided to our knowledge, the first rigorous justification of the Abrikosov lattice in this regime: at least the hexagonal lattice is the best among perfect lattice configurations (it beats the square lattice for example). Note that analogous results were obtained in [4, 1] but for the regime of $h_{\rm ex}$ very near H_{c_2} which is a very different regime where the problem becomes essentially linear.

These results are obtained once more by upper bounds on the minimal energy obtained via an explicit construction, combined with matching and ansatz-free lower bounds, made possible by the energy-splitting presented above and by the ergodic theorem approach (which allows in particular to control the number of vortices per unit volume after blow-up, around most blow-up centers).

References

[1] Aftalion, A; Serfaty, S. Lowest Landau level approach in superconductivity for the Abrikosov lattice close to H_{c_2}, *Selecta Math.* **13**, (2007), No 2, 183–202.

[2] Alberti, G.; Müller, S. A new approach to variational problems with multiple scales. *Comm. Pure Appl. Math.* **54**, (2001), No 7, 761-825.

[3] Almog, Y. On the bifurcation and stability of periodic solutions of the Ginzburg-Landau equations in the plane. *SIAM J. Appl. Math.* **61** (2000), no. 1, 149–171.

[4] Almog, Y., Abrikosov lattices in finite domains, *Comm. Math. Phys.* **262**, (2006), No. 3, 677–702.

[5] Berger, M. S.; Chen, Y. Y. Symmetric vortices for the nonlinear Ginzburg-Landau of superconductivity, and the nonlinear desingularization phenomenon. *J. Funct. Anal.* **82** (1989), no. 2, 259–295.

[6] Bethuel, F.; Brezis, H.; Hélein, F. *Ginzburg-Landau vortices.* Birkhäuser Boston, Boston, 1994.

[7] Cassels, On a problem of Rankin about the Epstein zeta-function *Proc. Glasgow Math. Assoc,* (1959).

[8] Chapman, S. J. Nucleation of superconductivity in decreasing fields. *Eur. J. Appl. Math.* **5** (1994), part 1, 449–468; part 2, 468–494.

[9] Chapman, S. J.; Rubinstein, J.; Schatzman, M. A mean-field model of superconducting vortices. *Eur. J. Appl. Math.* **7** (1996), no. 2, 97–111.

[10] DeGennes, P. G. *Superconductivity of metal and alloys.* Benjamin, New York and Amsterdam, 1966.

[11] Gueron, S.; Shafrir, I. On a Discrete Variational Problem Involving Interacting Particles. *SIAM J. Appl. Math.* **60** (2000), no. 1, 1–17.

[12] Jaffe, A.; Taubes, C. *Vortices and monopoles.* Progress in Physics, 2. Birkhaüser, Boston, 1980.

[13] Jerrard, R. L.; Soner, H. M. The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differential Equations* **14** (2002), no. 2, 151–191.

[14] Manton, N.; Sutcliffe, P. *Topological solitons.* Cambridge monographs on mathematical physics. Cambridge University Press, 2004.

[15] Plohr, B. The existence, regularity, and behaviour at infinity of isotropic solutions of classical gauge field theories. Doctoral Dissertation, Princeton University, 1980.

[16] Plohr, B. The Behavior at Infinity of Isotropic Vortices and Monopoles B. Plohr, *J. Math. Phys.*, **22** (1981), 2184–2190.

[17] Rankin, A. minimum problem for the Epstein zeta function, *Proc. Glasgow Math. Assoc* **1**, (1953). 149–158.

[18] Saint-James, D.; Sarma, G.; Thomas, E. J. *Type-II superconductivity.* Pergamon Press, Oxford, 1969.

[19] Sandier, E; Serfaty, S. *Vortices in the Magnetic Ginzburg-Landau Model*, Birkhaüser, 2007.

[20] Sandier, E.; Serfaty, S. On the energy of type-II superconductors in the mixed phase. *Rev. Math. Phys.* **12** (2000), no. 9, 1219–1257.

[21] Sandier, E.; Serfaty, S. A rigorous derivation of a free-boundary problem arising in superconductivity. *Ann. Sci. École Norm. Sup. (4)* **33** (2000), no. 4, 561– 592.

[22] Sandier, E.; Serfaty, S., From the Ginzburg-Landau Model to Vortex Lattice Problems, to appear.

[23] Serfaty, S. Local minimizers for the Ginzburg-Landau energy near critical magnetic field, part I. *Comm. Contemp. Math.* **1** (1999), no. 2, 213–254; part II, 295–333.

[24] Serfaty, S. Stable configurations in superconductivity: uniqueness, multiplicity and vortex-nucleation. *Arch. Ration. Mech. Anal.* **149** (1999), 329–365.

[25] Tinkham, M. *Introduction to superconductivity.* Second edition. McGraw-Hill, New York, 1996.