Compactness, kinetic formulation, and entropies for a problem related to micromagnetics

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Abstract

We carry on the study of [RS] on the asymptotics of a family of energy-functionals related to micromagnetics. We prove compactness for families of uniformly bounded energies releasing the LBP condition we had previously set. Such families converge to unit-valued divergence-free vector-fields that are tangent to the boundary of the domain, and we found in [RS] that the energy-functionals Γ-converge to a limiting jump-energy of such configurations. We examine the behavior of certain truncated fields which serve to construct "entropies", and to provide an improved lower bound. We give a kinetic formulation of the problem, and show that the limiting divergence-free problem is supplemented, in the case of minimizers, with a sign condition which can in turn, using the kinetic formulation, be interpreted as an entropy condition that should play a role in uniqueness questions.

key-words: micromagnetics, Γ-convergence, compensated-compactness, entropies, kinetic equations.

I Introduction

In this paper, we carry on the study started in our previous paper [RS], on the following energy-functional, related to micromagnetics:

$$E_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H_u|^2. \tag{I.1}$$

Here, Ω is a bounded simply connected domain of $\mathbb{R}^2$, u is a unit-valued vector-field (corresponding to the magnetization) in $H^1(\Omega, S^1)$, and H_u, the demagnetizing field created by u (non-local term in u), is given by

$$\begin{cases} \text{div } (\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^2 \\ \text{curl } H_u = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{I.2}$$

where $\tilde{u}$ is the extension of u by 0 in $\mathbb{R}^2 \backslash \Omega$. For a general presentation and motivations of this study, we refer to [RS] and all the references therein.

We are interested in the asymptotics as $\varepsilon \to 0$ of families of uniformly bounded energy: $E_\varepsilon(u_\varepsilon) \leq C$. For such families, we denote for simplicity by H_ε the demagnetizing field associated to u_ε, and we recall that $\int_\Omega |H_\varepsilon|^2 \leq C\varepsilon$ and in fact $H_\varepsilon \to 0$ in $\cap_{q<\infty} L^q(\mathbb{R}^2)$.

One of the main questions on this problem was to know whether the condition $|u_\varepsilon| = 1$ passes to the limit i.e. get L^q compactness on such u_ε. We proved such compactness in [RS] under the LBP condition ("locally bounded phase condition"). Here, we are now able to release this condition and replace it by a much simpler assumption. More specifically, $u_\varepsilon \in H^1(\Omega, S^1)$ has a lifting $\varphi_\varepsilon \in H^1(\Omega, \mathbb{R})$ (see [BZ]) such that $u_\varepsilon = e^{i\varphi_\varepsilon}$ a.e. Under the condition that u_ε admits such a lifting remaining bounded in L^∞, we prove L^q compactness of φ_ε and u_ε, by adjusting the arguments we used in [RS] (see Proposition II.1). Then, denoting by u and φ the limits, we recall that passing to the limit in (I.2) yields

$$\text{(I.3)} \qquad \begin{cases} \text{div } \tilde{u} = 0 & \text{in } \mathbb{R}^2 \\ |\tilde{u}| = 1 & \text{in } \Omega, \end{cases}$$

equivalent to

$$\text{(I.4)} \qquad \begin{cases} \text{div } u = 0 & \text{in } \Omega \\ u \cdot \nu = 0 & \text{on } \partial\Omega \\ |\tilde{u}| = 1 & \text{in } \Omega, \end{cases}$$

thus the limiting fields lie among unit-valued divergence-free fields tangent to the boundary. Such fields always have singularities, typically line singularities. For such a u, we can always find a Lipschitz function g such that

$$\text{(I.5)} \qquad \begin{cases} u = \nabla^\perp g = (-\partial_{x_2} g, \partial_{x_1} g) & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega \\ |\nabla g| = 1 & \text{in } \Omega. \end{cases}$$

Thus, g is solution of an eikonal equation, and the question was also to understand which solutions of this eikonal equations are selected through this limiting process.
We also recall one of the main observations of [RS] was that we could write

$$\begin{aligned} C \geq E_\varepsilon(u_\varepsilon) &= \int_\Omega \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H_\varepsilon|^2 \\ \text{(I.6)} \qquad &\geq 2\int_\Omega |\nabla\varphi_\varepsilon||H_\varepsilon| \geq 2\int_\Omega |\nabla\varphi_\varepsilon \cdot H_\varepsilon|. \end{aligned}$$

But thanks to (I.2),

$$\text{(I.7)} \qquad \nabla\varphi_\varepsilon \cdot H_\varepsilon = \nabla\varphi_\varepsilon \cdot (H_\varepsilon + u_\varepsilon) - \nabla\varphi_\varepsilon \cdot u_\varepsilon = \text{div } (\varphi_\varepsilon(u_\varepsilon + H_\varepsilon) + u_\varepsilon^\perp),$$

where $u^\perp$ denotes $(-u_2, u_1)$. Hence, the quantity $\mu_\varepsilon = \nabla\varphi_\varepsilon \cdot H_\varepsilon = \text{div } (\varphi_\varepsilon(u_\varepsilon + H_\varepsilon) + u_\varepsilon^\perp)$ remains bounded in $L^1(\Omega)$ and we proved that it converges weakly in the sense of measures to the bounded Radon measure $\mu_{u,\varphi}$ defined by

$$\mu_{u,\varphi} := \text{div } (\varphi u + u^\perp).$$

Thus the limit (u, φ) belongs to the class $\mathcal{C}$ which we had defined as

Definition I.1 *$\mathcal{C}$ is the class of couples (u, φ) such that*
1) $u : \Omega \to S^1$
2) $\text{div } \tilde{u} = 0$ in $\mathcal{D}'(\mathbb{R}^2)$
3) $\varphi \in L^1(\Omega, \mathbb{R})$ and $u = e^{i\varphi}$ a.e. in Ω
4) $\mu_{u,\varphi} := \text{div } (\varphi u + u^\perp)$ is a bounded Radon measure on Ω.

We denoted by $\|\mu_{u,\varphi}\|$ its total mass and $(u, \varphi) \mapsto 2\|\mu_{u,\varphi}\|$ was the "Γ-limit" of the family E_ε. We explained that $\mu_{u,\varphi}$ is supported on the singular set of the limiting φ (it is 0 wherever φ is C^1) and carries a jump cost along the singular lines, and we proved that the minimum of $\|\mu_{u,\varphi}\|$ over $\mathcal{C}$ is $|\partial\Omega|$, the perimeter of Ω, achieved in particular by $u_* = \nabla^\perp \text{dist}(., \partial\Omega)$. Conversely, we proved (see [RS] Theorems 1.3 and 1.7) that we can construct a sequence $u_\varepsilon \to u_*$ such that $E_\varepsilon(u_\varepsilon) \to 2|\partial\Omega|$ (at least when $\partial\Omega$ is a finite union of analytic curves). We had conjectured that u_* and $-u_*$ were the only minimizers of $\|\mu_{u,\varphi}\|$ thus the ones selected by the minimization of E_ε.

In Section II, in addition to improving the compactness result, we introduce the truncated fields, already used in [RS], defined by

$$\text{(I.8)} \qquad \begin{cases} T^a\varphi := inf(\varphi, a) \\ T^a u := e^{iT^a\varphi}. \end{cases}$$

We prove that, at the limit, not only $\mu_{u,\varphi}$ is a bounded Radon measure, but also div $T^a u$ seen as a function of (x, a) is a bounded Radon measure on $\Omega \times \mathbb{R}$, with $\mu_{u,\varphi} = -\int_{\mathbb{R}} \text{div } T^a u \, da$ (see Theorem 1). This condition turns out to be a better one to define a limiting class than belonging to $\mathcal{C}$. The energy is bounded below as follows :

$$\text{(I.9)} \qquad \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq 2 \int_{\mathbb{R} \times \Omega} |\text{div } T^a u| \, da \, dx \geq 2 \int_\Omega |\mu_{u,\varphi}|,$$

which is indeed a finer lower bound than $2\|\mu_{u,\varphi}\|$, i.e. the last inequality can be strict. This can be seen through the BV case : if we have the additional assumption that φ (and thus u) is in $BV(\Omega)$ (φ then has a "jump set" S), this lower bound can be expressed explicitely with the formula

$$\text{(I.10)} \qquad \int_{\mathbb{R} \times \Omega} |\text{div } T^a u| \, da \, dx = \int_S w(X)$$

where X is the half-jump of φ along its jump set S, and w is a certain positive function equal to $2(\sin X - X \cos X)$ on $[0, \pi]$, and extended explicitely on the whole of $\mathbb{R}^+$ (see Corollary 1). This result extends the formula obtained in [RS], Theorem 5, which was restricted to $X \in [0, \pi)$. Also for $X > \pi$, we can notice that this lower bound becomes strictly better than $2\|\mu_{u,\varphi}\|$ (see Remark II.2). This already shows one interest of introducing the quantities div $T^a u$.

In Section III, we give a kinetic interpretation of the problem. This idea was used for a close problem in [JP] and initially introduced in [LPT]. Setting

$$\chi(x, a) := \mathbf{1}_{\varphi(x) \leq a},$$

we show that χ satisfies the "kinetic equation"

$$\text{div}_x(\chi(e^{ia})^\perp) = \nabla\chi \cdot (e^{ia})^\perp = -\partial_a(\text{div } T^a u). \tag{I.11}$$

Here, observe that the truncated fields $T^a u$ appear naturally in this formulation, and also that $\chi(e^{ia})^\perp$ corresponds to the "entropies" $\Phi_e(u)$ used in [DKMO]. This kinetic interpretation, which has the advantage of being very simple for our problem, allows, as in [JP], to get another proof of the compactness of φ_ε and u_ε (which, this time, relies on kinetic averaging lemmas) and to get improved Sobolev regularity for the limit φ (see Proposition III.1). Should one expect BV regularity at the limit? In the similar "Aviles-Giga problem" which was the one studied by [ADM, DKMO, JK, JP], where the constraint $|u| = 1$ is released and replaced by the constraint that u is divergence-free, it was shown in [ADM] that there exist configurations in the limiting "Aviles-Giga space" which are not in BV. Yet, they might not be achieved as limits of configurations of bounded energy, or the question remains open whether the total Γ-limit set fills $AG_e(\Omega)$ or not, and whether or not it is included in $BV(\Omega)$. The question is identical in our case with AG_e replaced by $\mathcal{C}$ or by the subclass $\int_{\mathbb{R}\times\Omega} |\text{div } T^a u|\, dx\, da < \infty$. Let us mention that, after this work was completed, this kinetic formulation (I.11) was used in [LR] to prove that configurations with vanishing div $T^a u$ are $H^{\frac{1}{2}}$, and Lipschitz except at a finite number of points (which cannot be the case only assuming div $(\varphi u + u^\perp) = 0$); also related regularity results were proved in [JOP].

In Section IV, we give additional properties for almost minimizing sequences i.e. sequences such that

$$E_\varepsilon(u_\varepsilon) \to 2 \min \|\mu_{u,\varphi}\| = 2|\partial\Omega|.$$

Going back to (I.6), this fact implies that the negative (or positive) part of μ_ε tends to 0, and at the limit $\mu_{u,\varphi} \geq 0$ or $\mu_{u,\varphi} \leq 0$. Thus, changing u to $-u$ if necessary, minimizers converge to u satisfying the two conditions:

$$\begin{cases} \text{div } u = 0 & \text{in } \Omega \\ u \cdot \nu = 0 & \text{on } \partial\Omega \\ \mu_{u,\varphi} = \text{div } (\varphi u + u^\perp) \geq 0 & \text{in } \Omega \end{cases} \tag{I.12}$$

The sign condition for the measure can be reinterpreted in the light of the truncated fields $T^a u$: we prove, using the co-area formula, that

$$\forall a \in \mathbb{R} \quad \text{div } T^a u \leq 0 \tag{I.13}$$

which decomposes the sign condition $\mu_{u,\varphi} \geq 0$ (see Theorem 2). Now this relation (I.13) can be seen as an entropy sign condition for the equation

$$\begin{cases} \text{div } u = 0 \\ |u| = 1 \end{cases} \tag{I.14}$$

which itself can be seen as a scalar conservation law, as it was pointed out in [DKMO]. Indeed, if u and φ solving (I.14) are regular enough, div $T^a u$ vanishes identically for all

$a \in \mathbb{R}$, thus the truncations T^a can be considered as "entropies" in that sense. This entropy sign condition (I.13) can also be written in integral form:

$$\text{(I.15)} \qquad \forall f \in C^1(\mathbb{R}) \text{ such that } f' \geq 0 \text{ and } f \in L^1(\mathbb{R}^+), \quad \text{div } (F(\varphi), G(\varphi)) \geq 0,$$

where

$$F(t) = -\int^t f(s) \sin s \, ds \qquad G(t) = \int^t f(s) \cos s \, ds.$$

Condition (I.15) or (I.13) thus supplements (I.4) with an entropy-type constraint that could allow to get uniqueness, as entropies do for scalar conservation laws. Then, u would be equal to $u_* = \nabla^\perp \text{dist}(., \partial\Omega)$ the viscosity solution of (I.5), as we conjectured in [RS] (the conjecture was supported by a heuristical argument that could be made rigorous in the BV case). This question also arises in the kinetic version: knowing that

$$\nabla\chi \cdot (e^{ia})^\perp = -\partial_a(\text{div } T^a u), \qquad \text{div } T^a u \leq 0$$

and u being prescribed on $\partial\Omega$, does it imply uniqueness? Looking for it in this formulation is natural in view of uniqueness results for similar time-dependent scalar conservation laws proved by B. Perthame in [Pe]. Yet, at this stage of development of this method, the uniqueness does not seem to follow straightforwardly from the aforementioned result.

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II Proof of compactness and lower bound

Here we improve the compactness result that we obtained in [RS]: we are able, by using the ingredients of [RS] (truncations, compensated-compactness and convexity) in a more efficient way, to give a self-contained proof of compactness, releasing the LBP condition and replacing it with the only condition that the lifting φ is bounded in L^∞.

Proposition II.1 *Let $\varepsilon_n \to 0$ and $u_n \in H^1(\Omega, S^1)$ with a lifting $\varphi_n \in H^1(\Omega, \mathbb{R})$ i.e. such that $u_n = e^{i\varphi_n}$ a.e., and assume*

$$\text{(II.1)} \qquad E_{\varepsilon_n}(u_n) \leq C$$

$$\text{(II.2)} \qquad \|\varphi_n\|_{L^\infty(\Omega)} \leq N.$$

Then, up to extraction, there exist u and φ in $\cap_{q<\infty} L^q(\Omega)$ such that

$$\varphi_n \to \varphi \text{ and } u_n \to u \text{ in } \cap_q L^q(\Omega).$$

Proof: Since u_n and φ_n are bounded in L^∞, extracting a subsequence if necessary, we can assume that they converge to u and φ weakly-* in L^∞. Moreover,

$$\text{(II.3)} \qquad C \geq E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\nabla\varphi_n \cdot H_n| = 2\int_\Omega |\text{div } (\varphi_n(u_n + H_n) + u_n^\perp)|.$$

-*Step 1:* Let p be a fixed integer. As in the proof of Lemma 4.5 of [RS], using the co-area formula, which applies since $\varphi_n \in H^1(\Omega) \subset BV(\Omega)$, we have

$$\int_{p\pi \leq \varphi_n \leq p\pi + \frac{\pi}{4}} |\nabla \varphi_n \cdot H_n| = \int_0^{\frac{\pi}{4}} d\eta \int_{\{p\pi+\eta=\varphi_n\}} |H_n \cdot \nu|, \tag{II.4}$$

where ν denotes the outer unit-normal. The left-hand side is bounded by (II.3), hence, using the mean-value theorem,

$$\exists \eta_{p,n} \in [0, \frac{\pi}{4}] \text{ such that } \quad \int_{\partial\{p\pi+\eta \leq \varphi_n\}} |H_n \cdot \nu| \leq C. \tag{II.5}$$

Then, we can define as in [RS] the truncated phases

$$T_p\varphi_n = \begin{cases} p\pi + \eta_{p,n} & \text{if } \varphi_n \leq p\pi + \eta_{p,n} \\ (p+1)\pi + \eta_{p+1,n} & \text{if } \varphi_n \geq (p+1)\pi + \eta_{p+1,n} \\ \varphi_n & \text{otherwise} \end{cases} \tag{II.6}$$

and

$$T_p u_n = e^{iT_p\varphi_n}. \tag{II.7}$$

Let $U_{p,n} = \{x \in \Omega, p\pi + \eta_{p,n} \leq \varphi_n \leq (p+1)\pi + \eta_{p+1,n}\}$. Observe that $T_p\varphi_n = \varphi_n$ in $U_{p,n}$ and $\nabla T_p\varphi_n = 0$ in $\Omega \backslash U_{p,k}$.

- *Step 2:* We prove that div $T_p u_n$ is compact in $H^{-1}(\Omega)$, as in the proof of Lemma 4.5 of [RS].
Let $\xi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_\Omega \xi \text{div } (T_p u_n) &= \int_{U_{p,n}} \xi (\text{div } u_n) \\ &= -\int_{U_{p,n}} \xi \text{div } H_n = \int_{U_{p,n}} H_n \cdot \nabla\xi - \int_{\partial U_{p,n}} \xi (H_n \cdot \nu). \end{aligned} \tag{II.8}$$

But,

$$\left| \int_{U_{p,n}} H_n \cdot \nabla\xi \right| \leq \|H_n\|_{L^q} \|\nabla\xi\|_{L^{q'}} \leq o(1) \|\nabla\xi\|_{L^{q'}} \qquad \forall q < \infty, \tag{II.9}$$

and, by construction (II.5),

$$\left| \int_{\partial U_{p,n}} \xi (H_n \cdot \nu) \right| \leq C \|\xi\|_{L^\infty}. \tag{II.10}$$

In view of (II.8), div $T_p u_n$, which is bounded in $W^{-1,q}(\Omega)$ for all $q < \infty$, is the sum of a term which is compact in $W^{-1,q}(\Omega)$ for all $q < \infty$ and a term which is bounded in the sense of measures. But, by Murat's theorem (see [Mu]), something bounded in the sense of measures and in $W^{-1,q}(\Omega)$ for all q is compact in $\cap_{q<\infty} W^{-1,q}(\Omega)$. Hence, we deduce the desired result.

- *Step 3:* We prove that div $(T_p\varphi_n(T_p u_n + H_n) + (T_p u_n)^\perp)$ is compact in $H^{-1}(\Omega)$. To do so, as in the proof of Lemma 4.5 of [RS], set

$$D_n = T_p\varphi_n(T_p u_n + H_n) + (T_p u_n)^\perp - T_p\varphi_n H_n(1-\chi_n),$$

where χ_n denotes the characteristic function of $U_{p,n}$, and

$$E_n = T_p\varphi_n H_n(1-\chi_n).$$

First, using the fact that $T_p\varphi_n = \varphi_n$ in $U_{p,n}$ and $\nabla T_p\varphi_n = 0$ in $\Omega\backslash U_{p,k}$, we have

$$\begin{aligned}\int_\Omega \xi \text{div } D_n &= \int_{U_{p,n}} \xi(\nabla\varphi_n\cdot u_n + \text{div } u_n^\perp) + \int_\Omega \xi T_p\varphi_n \text{div } (T_p u_n + H_n)\\ &\quad + \int_\Omega \xi\nabla T_p\varphi_n\cdot H_n + \int_{\Omega\backslash U_{p,n}} (\nabla\xi\cdot H_n)T_p\varphi_n\end{aligned}$$

The first term vanishes identically because div $u_n^\perp = -\nabla\varphi_n\cdot u_n$ for $\varphi_n\in H^1(\Omega)$. For the second term, we use div $(T_p u_n + H_n) = 0$ in $U_{p,n}$ and div H_n in $\Omega\backslash U_{p,n}$. Thus

$$\begin{aligned}\int_\Omega \xi \text{div } D_n &= \int_{\Omega\backslash U_{p,n}} \xi T_p\varphi_n \text{div } H_n + T_p\varphi_n\nabla\xi\cdot H_n + \int_\Omega \xi\nabla T_p\varphi_n\cdot H_n\\ &= \int_{\Omega\backslash U_{p,n}} \text{div } (\xi H_n)T_p\varphi_n + \int_{U_{p,n}} \xi\nabla\varphi_n\cdot H_n\\ &= -\int_{\partial U_{p,n}} \xi T_p\varphi_n H_n\cdot\nu + \int_{U_{p,n}} \xi\nabla\varphi_n\cdot H_n.\end{aligned}$$

Then,

$$\left|\int_{U_{p,n}} \xi\nabla\varphi_n\cdot H_n\right| \le \|\xi\|_{L^\infty}\int_\Omega |\nabla\varphi_n\cdot H_n| \le C\|\xi\|_{L^\infty},$$

and from (II.5)

$$\int_{\partial U_{p,n}} |\xi T_p\varphi_n(H_n\cdot\nu)| = \int_{\{\varphi_n = p\pi+\eta_{p,n}\}\cup\{\varphi_n=(p+1)\pi+\eta_{p+1,n}\}} |\xi T_p\varphi_n H_n\cdot\nu| \le C\|\xi\|_{L^\infty},$$

where we have used the fact that ξ is compactly supported in Ω. Thus, we deduce that div D_k remains bounded in the sense of measures. Furthermore,

$$\left|\int_\Omega \xi \text{div } E_n\right| = \left|\int_{\Omega\backslash U_{p,k}} T_p\varphi_n\nabla\xi\cdot H_n\right| \le C\|\xi\|_{H_0^1}\|H_n\|_{L^2} \le o(1)\|\xi\|_{H_0^1},$$

hence div E_n tends to 0 in $H^{-1}(\Omega)$. As in the previous step, we can conclude that div $(D_n + E_n)$ is compact in $H^{-1}(\Omega)$, which is the desired result.

- *Step 4:* Let us now consider the truncated field $T_p u_n$. Its lifting $T_p\varphi_n$ takes its values in the interval $[p\pi, (p+1)\pi + \frac{\pi}{4}]$ of length $\frac{5\pi}{4}$. This is enough to deduce compactness of $T_p\varphi_n$ in $\cap_{q<\infty} L^q$ as in [RS]. We will write the argument again.
We can find a measurable $l_0(x) \in [p\pi, (p+2)\pi]$ such that $T_p u_n$ converges weakly to some $\alpha(x)e^{il_0(x)}$ (identifying $\mathbb{R}^2$ with $\mathbb{C}$) where $\alpha \geq 0$. Then, we denote by (β, γ) the weak limit of $e^{-il_0} T_p\varphi_n T_p u_n$. Following [RS], we define

$$\text{(II.11)} \qquad \begin{cases} A_n = (T_p u_n)^\perp \\ B_n = T_p\varphi_n T_p u_n + (T_p u_n)^\perp \\ C_n = T_p\varphi_n T_p u_n \end{cases}$$

We have

$$\text{(II.12)} \qquad \begin{cases} A_n \rightharpoonup i\alpha e^{il_0} = A \\ C_n \rightharpoonup (\beta + i\gamma)e^{il_0} \\ B_n \rightharpoonup (\beta + i(\gamma + \alpha))e^{il_0} = B \end{cases}$$

Then, we can apply the compensated-compactness lemma of Murat and Tartar: $A_n \rightharpoonup A$ in $L^2(\Omega)$, curl A_n is compact in $H^{-1}(\Omega)$ by Step 2, $B_n \rightharpoonup B$ in $L^2(\Omega)$ and div B_n is compact in $H^{-1}(\Omega)$ by Step 3, hence

$$1 = A_n \cdot B_n \rightharpoonup A \cdot B = \alpha(\gamma + \alpha).$$

- *Step 5:* Let us now introduce ν_x the Young measure generated by $T_p\varphi_n$. It is supported in $[p\pi, (p+1) + \frac{\pi}{4}]$ hence in $[l_0 - 2\pi, l_0 + 2\pi]$. By definition of Young measures and uniqueness of the weak limit,

$$\text{(II.13)} \qquad \int d\nu_x = 1$$

$$\text{(II.14)} \qquad \alpha \cos l_0 = \int \cos t \, d\nu_x(t) \quad \text{a.e.}$$

$$\text{(II.15)} \qquad \alpha \sin l_0 = \int \sin t \, d\nu_x(t) \quad \text{a.e.}$$

$$\text{(II.16)} \qquad \beta \cos l_0 - \gamma \sin l_0 = \int t \cos t \, d\nu_x(t) \quad \text{a.e.}$$

$$\text{(II.17)} \qquad \beta \sin l_0 + \gamma \cos l_0 = \int t \sin t \, d\nu_x(t) \quad \text{a.e.}$$

It is easy to check that

$$\text{(II.18)} \qquad \alpha = \int \cos(t-l_0)d\nu_x(t) = \int_{-2\pi}^{2\pi} \cos t \, d\nu_x(t+l_0)$$

$$\text{(II.19)} \qquad 0 = \int \sin(t-l_0)d\nu_x(t)$$

$$\text{(II.20)} \qquad \gamma = \int t\sin(t-l_0)d\nu_x(t) = \int_{-2\pi}^{2\pi} t\sin t \, d\nu_x(t+l_0).$$

In [RS] we pointed out that in $[-2\pi, 2\pi]$, $t\sin t + 2\cos t \leq 2$, which implies, integrating against $d\nu_x(.+l_0)$, that $\gamma \leq 2-2\alpha$ a.e. Following [RS] again, this fact combined with $\alpha(\gamma+\alpha)=1$ a.e. implies that $\alpha = 1$ a.e. and in view of (II.18) that ν_x is supported in $\{l_0-2\pi, l_0, l_0+2\pi\}$ a.e. But ν_x is also supported in an interval of length $< 2\pi$ hence its support has to be reduced to a point and ν_x is a Dirac mass at almost every point of Ω. We can conclude that $T_p\varphi_n$ converges a.e. to its weak limit, hence converges strongly in $\cap_{q<\infty}L^q(\Omega)$ by Lebesgue's theorem.

- *Step 6:* We observe that

$$\sum_{p=-P}^{P} T_p\varphi_n = T_{-\pi P+\eta_{-P,n}}^{\pi(P+1)+\eta_{P+1,n}}\varphi_n + \sum_{p=-P}^{P}\eta_{p,n} + \pi P,$$

where

$$T_a^b\varphi = \begin{cases} a & \text{if } \varphi \leq a \\ b & \text{if } \varphi \geq b \\ \varphi & \text{if } \varphi \in [a,b]. \end{cases}$$

But P being set, $\sum_{p=-P}^{P}\eta_{p,n}$ is compact (when $n\to\infty$) and from the result of Step 5 (which was true for any p), $\sum_{p=-P}^{P}T_p\varphi_n$ is compact in $\cap_{q<\infty}L^q(\Omega)$, hence $T_{-\pi P+\eta_{-P,n}}^{\pi(P+1)+\eta_{P+1,n}}\varphi_n$ is also compact in $\cap_{q<\infty}L^q$. If P is chosen large enough compared to N (such that $\|\varphi_n\|_{L^\infty} \leq N$), then

$$\forall n, \qquad T_{-\pi P+\eta_{-P,n}}^{\pi(P+1)+\eta_{P+1,n}}\varphi_n = \varphi_n$$

and we deduce that φ_n is compact in $\cap_{q<\infty}L^q(\Omega)$, and u_n too. □

We conclude by the following convergence and lower bound result.

Theorem 1 *Let $\varepsilon_n \to 0$ and $u_n \in H^1(\Omega, S^1)$ with a lifting $\varphi_n \in H^1(\Omega,\mathbb{R})$ i.e. $u_n = e^{i\varphi_n}$ a.e., and such that*

$$\text{(II.21)} \qquad E_{\varepsilon_n}(u_n) \leq C$$

$$\text{(II.22)} \qquad \|\varphi_n\|_{L^\infty(\Omega)} \leq N.$$

Then, up to extraction, there exists u and φ in $\cap_{q<\infty} L^q(\Omega)$ such that

$$\begin{aligned} \varphi_n &\to \varphi \ \textit{in} \ \cap_{q<\infty} L^q(\Omega) \\ u_n &\to u \ \textit{in} \ \cap_{q<\infty} L^q(\Omega). \end{aligned}$$

Moreover, $(u, \varphi) \in \mathcal{C}$ and

$$\liminf_{n\to\infty} E_{\varepsilon_n}(u_n) \geq 2\|\mu_{u,\varphi}\| \geq 2|\partial\Omega|, \tag{II.23}$$

and div $T^a u$ *is a bounded Radon measure on $\Omega \times \mathbb{R}$, with $a \mapsto$* div $T^a u$ *continuous from $\mathbb{R}$ to $\mathcal{D}'(\Omega)$. In addition,*

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}} |\mathrm{div}\ T^a u| dx\, da \leq \liminf_{n\to\infty} \int_\Omega |\nabla\varphi_n \cdot H_n| \leq \liminf_{n\to\infty} \frac{E_{\varepsilon_n}(u_n)}{2} < \infty \\ &-\int_{\mathbb{R}} \mathrm{div}\ T^a u\, da = \mu_{u,\varphi} \quad \textit{in } \mathcal{D}'(\Omega). \end{aligned}$$

Proof : The first part was proved in Proposition II.1, (II.23) was proved in [RS]. We only need to prove the last assertion.
Let $f \in C_0^\infty(\mathbb{R})$, $\xi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \left| \int_\Omega f(\varphi_n)\xi(x)\nabla\varphi_n \cdot H_n \right| &\leq \|f\|_{L^\infty}\|\xi\|_{L^\infty} \int_\Omega |\nabla\varphi_n \cdot H_n| \\ &\leq \frac{1}{2} E_{\varepsilon_n}(u_n)\|f\|_{L^\infty}\|\xi\|_{L^\infty}. \end{aligned} \tag{II.24}$$

We then use the co-area formula as we did in (II.4),

$$\begin{aligned} \int_\Omega f(\varphi_n)\xi(x)\nabla\varphi_n \cdot H_n &= \int_{\mathbb{R}} f(a) \left(\int_{\varphi_n(x)=a} (H_n \cdot \nu)\xi(x) dx \right) da, \\ &= \int_{\mathbb{R}} f(a) \left(\int_{\partial\{\varphi_n(x)\leq a\}} (H_n \cdot \nu)\xi(x) dx \right) da, \end{aligned} \tag{II.25}$$

because ξ is compactly supported in Ω, where ν denotes the outer unit-normal to the level-set $\{\varphi_n(x) \leq a\}$. Thus, integrating by parts and using the relation div $H_n = -$div u_n, we are led to

$$\int_\Omega f(\varphi_n)\xi(x)\nabla\varphi_n \cdot H_n = \int_{\mathbb{R}} f(a) \left(\int_{\varphi_n(x)\leq a} (-\xi(x)\mathrm{div}\ u_n + H_n \cdot \nabla\xi) dx \right) da. \tag{II.26}$$

Let us now introduce

$$\chi_n(x, a) = \mathbf{1}_{\varphi_n(x)\leq a}. \tag{II.27}$$

We observe that

$$\chi_n \mathrm{div}\ u_n = \mathrm{div}\ T^a u_n, \tag{II.28}$$

hence

$$
\begin{aligned}
\int_\Omega f(\varphi_n)\xi(x)\nabla\varphi_n\cdot H_n &= \int_{\mathbb{R}} f(a)\left(-\int_\Omega \xi(x)\mathrm{div}\ T^a u_n\,dx + \int_\Omega H_n\chi_n\cdot\nabla\xi\,dx\right)da \\
&= -\int_{\Omega\times\mathbb{R}} f(a)\xi(x)\mathrm{div}\ (T^a u_n + \chi_n H_n)\,dx\,da. \qquad \text{(II.29)}
\end{aligned}
$$

We deduce from (II.29) and (II.24) that div $(T^a u_n + \chi_n H_n)$ remains bounded by $\int_\Omega |\nabla\varphi_n\cdot H_n|$ in the sense of measures in $\Omega\times\mathbb{R}$. In fact

$$
\mathrm{div}\ (T^a u_n + \chi_n H_n) = -(\nabla\varphi_n\cdot H_n)\delta_{\varphi_n(x)=a}. \qquad \text{(II.30)}
$$

But $\chi_n H_n \to 0$ in $\cap_{q<\infty} L^q(\Omega)$, hence div $(\chi_n H_n)\to 0$ in $W^{-1,q}(\Omega)$. On the other hand, from the compactness result (Proposition II.1), since $u_n\to u$ in $\cap_{q<\infty}L^q(\Omega)$, we also have (by continuity of T^a) that $T^a u_n \to T^a u$ in $\cap_{q<\infty}L^q(\Omega)$, independently of a, thus div $T^a u \rightharpoonup$ div $T^a u$ in $W^{-1,q}(\Omega\times\mathbb{R})$ and we conclude that the weak limit of div $(T^a u_n + \chi_n H_n)$ is div $T^a u$ and is a bounded measure on $\Omega\times\mathbb{R}$ with

$$
\int_{\mathbb{R}}\int_\Omega |\mathrm{div}\ T^a u|\,dx\,da \le \liminf_{n\to\infty}\int_\Omega |\nabla\varphi_n\cdot H_n|.
$$

In addition, taking $f=1$ in (II.29), we find that

$$
\nabla\varphi_n\cdot H_n = -\int_{\mathbb{R}} \mathrm{div}\ (T^a u_n + \chi_n H_n)\,da
$$

and passing to the limit when $n\to\infty$

$$
-\int_{\mathbb{R}} \mathrm{div}\ T^a u = \lim_{n\to\infty}\mu_n = \mu_{u,\varphi}.
$$

The proof of the continity of $a\mapsto$ div $T^a u$ is postponed until the end of Section IV. □

If we assume in addition that $\varphi\in BV(\Omega)$ (hence $u\in BV(\Omega)$ too by composition), by the structure theorem of BV functions, $D\varphi$ is a Radon measure, which can be split into three mutually singular parts

$$
D\varphi = \nabla\varphi\mathcal{L}^2 + (\varphi^+ - \varphi^-)\otimes n\mathcal{H}^1\lfloor_{S_\varphi} + D_c\varphi \qquad \text{(II.31)}
$$

where $\mathcal{L}^2$ is the Lebesgue measure, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, $D_c\varphi$ is the Cantor part of $D\varphi$, S_φ is the jump set of φ, n is the normal to S_φ pointing from S_φ into the + half-space, and φ^+ and φ^- are the approximate limits of φ on both "sides", + and − of S_φ. Similarly, u has a jump set $S_u\subset S_\varphi$ (but there is not necessarily equality since φ can jump by an integer multiple of 2π, in which case u does not jump), and has traces u_+, u_- on both sides of S_u. Since u is divergence-free, its normal component is preserved along the jump-set, i.e. $u_+\cdot n = u_-\cdot n$ along S_u, while along $S_\varphi\backslash S_u$, $u_+ = u_-$, and we denote in that case by θ the geometric angle $\in [0,\pi)$ between u and the normal n. One can define the half-jump of φ

$$X = \frac{|\varphi_+ - \varphi_-|}{2} \qquad \text{along } S_u. \tag{II.32}$$

Pointwise, there exists unique $X' \in (0,\pi)$ and $k \in \mathbb{N}$ such that

$$X = X' + k\pi. \tag{II.33}$$

We can then define on $\mathbb{R}^+$ the continuous function w as follows:

$$w(X) = h(X') + k\tilde{h}(X'), \tag{II.34}$$

where

$$\begin{aligned} h(t) &= 2(\sin t - t\cos t) \\ \tilde{h}(t) &= h(t) + h(\pi - t). \end{aligned}$$

(Observe that $4 \leq \tilde{h} \leq 2\pi$ on $[0,\pi]$.) As a by-product of Theorem 1, we get

Corollary 1 *If in addition $\varphi \in BV(\Omega)$, then,*

$$\int_{\mathbb{R}} |\mathrm{div}\ T^a u|\, da\lfloor_{S_u} = w(X)\mathcal{H}^1\lfloor_{S_u},$$

and,

$$\int_{\mathbb{R}} |\mathrm{div}\ T^a u|\, da\lfloor_{S_\varphi\setminus S_u} = k\tilde{h}(\theta)\mathcal{H}^1\lfloor_{S_\varphi\setminus S_u},$$

hence,

$$\liminf_{n\to\infty} \frac{E_{\varepsilon_n}(u_n)}{2} \geq \int_{S_u} w(X) + \int_{S_\varphi\setminus S_u} k\tilde{h}(\theta). \tag{II.35}$$

Proof: One can show as in [RS], proof of Theorem 5, that in the BV sense,

$$|\mathrm{div}\ T^a u| = |e^{ia}\cdot n - u_-\cdot n|\mathcal{H}^1\lfloor_{S_\varphi} \otimes da\lfloor_{[\varphi_-,\varphi_+]} \tag{II.36}$$

(changing n to $-n$ if necessary, we may assume that $\varphi_- \leq \varphi_+$). We may also work, poinwise, in the orthonormal frame (τ, n), with $\tau = -n^\perp$, so that the condition $u_-\cdot n = u_+\cdot n$ along S_u rewrites as $\sin\varphi_- = \sin\varphi_+$. We first consider the case $\varphi_- \in [0,\frac{\pi}{2}]$. We have

$$\varphi_+ = \varphi_- + 2X' + 2k\pi \tag{II.37}$$

with

$$\varphi_- + 2X' = \pi - \varphi_-. \tag{II.38}$$

We then only need to compute

$$\int_{\varphi_-}^{\varphi^+} |e^{ia}\cdot n - u_-\cdot n|\,da = \int_{\varphi_-}^{\varphi^+} |\sin a - \sin\varphi_-|\,da.$$

On S_u, where we have $\sin\varphi_+ = \sin\varphi_-$,

$$\begin{aligned}\int_{\varphi_-}^{\varphi^+} |\sin a - \sin\varphi_-|\,da &= (k+1)\int_{\varphi_-}^{\pi-\varphi_-} (\sin a - \sin\varphi_-)\,da + k\int_{\pi-\varphi_-}^{\varphi_-+2\pi} (\sin\varphi_- - \sin a)\,da\\ &= (k+1)(2\cos\varphi_- - 2X'\sin\varphi_-) + k\left((2\pi - 2X')\sin\varphi_- + 2\cos\varphi_-\right),\end{aligned}$$

where we have used the periodicity and (II.38). Then, using again (II.37) and (II.38), we have $\sin\varphi_- = \cos X'$ and $\cos\varphi_- = \sin X'$, hence

$$\int_{\varphi_-}^{\varphi^+} |e^{ia}\cdot n - u_-\cdot n|\,da = \int_{\varphi_-}^{\varphi^+} |\sin a - \sin\varphi_-|\,da = 2(k+1)h(X') + kh(\pi - X') = w(X).$$

The case where $\varphi_- \notin [0,\frac{\pi}{2}]$ can be treated similarly and yields the same formula. On $S_\varphi\backslash S_u$, we can consider that φ jumps from $\frac{\pi}{2}-\theta$ to $\frac{\pi}{2}-\theta+2k\pi$. In that case

$$\begin{aligned}\int_{\varphi_-}^{\varphi^+} |e^{ia}\cdot n - u_-\cdot n|\,da &= \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}-\theta+2k\pi} |\sin a - \cos\theta|\,da\\ &= k\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}+\theta} (\sin a - \cos\theta)\,da + k\int_{\frac{\pi}{2}+\theta}^{\frac{5\pi}{2}-\theta} (\cos\theta - \sin a)\,da\\ &= k(h(\theta) + h(\pi-\theta)) = k\tilde{h}(\theta).\end{aligned}$$

With (II.36), we get the conclusions of Corollary 1. □

Observe that h was already introduced in [RS], and is a positive increasing function from $[0,\pi]$ to $[0,2\pi]$. The formula (II.35) extends the one proved in [RS] for BV functions, which relied on computing $\int_\Omega |\mathrm{div}\,(\varphi u + u^\perp)|$ and was restricted to the case $X\in[0,\pi)$, i.e. unnecessary turns along the unit circle were excluded. Let us recall that in [RS], we also proved that there is no profile (solutions of the ODE associated to the minimization of E_ε) corresponding to jumps with $X>\pi$, i.e. to more than one turn on the unit circle.

If σ denotes the geometric half-angle ($\sigma\in[0,\frac{\pi}{2}]$) between u_+ and u_- (or shortest way to jump from u_+ to u_-), then $\sigma = X'$ if $X'\in[0,\frac{\pi}{2}]$, and $\sigma = \pi - X'$ if $X'\in[\frac{\pi}{2},\pi]$, and, examining the variations of the function h, one always has $h(X')\geq h(\sigma)$. Therefore, we deduce

Corollary 2

$$\liminf_{n\to\infty}\frac{E_{\varepsilon_n}(u_n)}{2} \geq \int_{S_u} 2(\sin\sigma - \sigma\cos\sigma),$$

where σ is the geometric half-angle between u_+ and u_-.

This provides a (weaker) lower bound which depends only on the limiting field u, and not on its lifting.

Remark II.1: $w(X) \geq h(X')$ and $w(X)$ vanishes if and only if the jump X is 0, hence a jump in φ always carries an energy cost. Additional turns on the unit circle carry additional costs.

Remark II.2: Corollary 1 proves that we can have

$$\int_{\mathbb{R}} |\text{div } T^a u|\, da > |\text{div } (\varphi u + u^\perp)|.$$

Indeed, if $\varphi \in BV$ and $X > \pi$, $X \notin \pi\mathbb{N}$,

$$|\text{div } (\varphi u + u^\perp)| = 2(\sin X - X \cos X) = h(X) < w(X).$$

For example, if $X > 0$ is a solution of the equation $tgX = X$, then $|\text{div } (\varphi u + u^\perp)| = 0$ while $w(X) > 0$.

As we mentioned, this remark proves that the lower bound obtained by inserting the quantities div $T^a u$ is finer in some cases than the lower bound $2\|\mu_{u,\varphi}\|$ derived in [RS] (which answers a question raised in [LR]).

III Kinetic interpretation

In this section, we give a kinetic interpration of the problem inspired from that used by Jabin and Perthame in [JP]. This allows to give an alternate proof of the compactness result and get extra Sobolev regularity for the limit. We will use the same notations as in the previous section.

The result relies on the following simple remarks. Since $\varphi_n \in H^1(\Omega) \subset BV(\Omega)$ and $\chi_n(.,a) = \mathbf{1}_{\varphi_n \leq a}$ is in $BV(\Omega)$, we can write, using the chain-rule,

(III.1) $$\nabla_x \chi_n(x,a) = -(\partial_a \chi_n(x,a))\nabla\varphi_n(x).$$

$\nabla_x \chi_n$ is clearly supported on the level-curve $\{x/\varphi_n(x) = a\}$, and $\partial_a \chi_n$ too. Therefore, since $u_n = e^{ia}$ on the support of $\nabla_x \chi_n$, we have

(III.2) $$\nabla\chi_n \cdot u_n^\perp = \nabla\chi_n \cdot (e^{ia})^\perp.$$

These relations (III.1) and (III.2) can be justified the following way: $\chi_n(x,a) = h(\varphi_n - a)$ where h is the Heaviside function; one replaces h by an affine approximation h_ε, takes $\chi_n^\varepsilon = h_\varepsilon(\varphi_n - a)$, then passes to the limit $\varepsilon \to 0$.

On the other hand, we recall that if φ_n and u_n are in $H^1(\Omega)$, we can write

(III.3) $$\text{div } u_n = \nabla\varphi_n \cdot u_n^\perp.$$

But, multiplying the relation (III.1) by $u_n^\perp$, we get

$$\nabla\chi_n \cdot u_n^\perp = -\partial_a\chi_n(x,a)\nabla\varphi_n \cdot u_n^\perp$$

hence in view of (III.2) and (III.3), we have

$$\nabla\chi_n \cdot (e^{ia})^\perp = -\partial_a\chi_n \text{div } u_n = -\partial_a(\chi_n \text{div } u_n).$$

Combining this with (II.28), we get the crucial relation

$$\nabla\chi_n \cdot (e^{ia})^\perp = -\partial_a(\text{div } T^a u_n), \tag{III.4}$$

which can be seen as a kinetic equation on χ_n, analogue of that obtained in [JP], for which the kinetic averaging lemmas apply.

Proposition III.1 *Under the same hypotheses as in Theorem 1, using (III.4) we find that up to extraction*

$$\varphi_n \to \varphi \textit{ in } \cap_{q<\infty} L^q(\Omega)$$
$$\chi_n(x,a) \to \chi(x,a) = \mathbf{1}_{\varphi(x)\leq a} \textit{ in } \cap_{q<\infty} L^q(\Omega),$$

and that at the limit $\varphi \in W^{s,p}(\Omega)$, $\forall s < \frac{1}{3}, p \leq \frac{5}{3}$, with

$$div_x(\chi(e^{ia})^\perp) = -\partial_a \text{div } T^a u \quad \textit{in } \Omega \times \mathbb{R}. \tag{III.5}$$

Proof: In view of (II.30)

$$\nabla\chi_n \cdot (e^{ia})^\perp = -\partial_a \text{div } T^a u_n = -\partial_a(G_n - \text{div } (\chi_n H_n))$$

with G_n bounded in $L^1(\Omega \times \mathbb{R})$, and div $(\chi_n H_n) \to 0$ in $\cap_{q<\infty} W^{-1,q}(\Omega)$. We are thus in a situation where the kinetic averaging lemma applies, for example as in [LPT] Theorem 3, using the version of [DLM], and we can conclude that

$$\exists p > 1, \quad \forall\psi \in C_0^\infty(\Omega), \int_{\mathbb{R}} \psi(a)\chi_n(a)da \text{ is compact in } L^p(\Omega).$$

Let Ψ be a primitive of ψ,

$$\int_{\mathbb{R}} \psi(a)\chi_n(a)da = \int_{\varphi_n(x)}^{+\infty} \psi(a)\, da = \Psi(+\infty) - \Psi(\varphi_n(x)).$$

Choosing $\psi = 1$ in $[-N, N]$ where N is such (by hypothesis) that $\forall n,\ \varphi_n \in [-N, N]$, we get that φ_n is compact in L^p hence in $\cap_{q<\infty} L^q(\Omega)$. Next, we can pass to the limit in (III.4). Since $\varphi_n \to \varphi$ in $L^q(\Omega)$, $\chi_n(x,a) \to \chi(x,a)$ in $\cap_q L^q(\Omega \times \mathbb{R})$. Thus

$$\nabla_x\chi_n \cdot (e^{ia})^\perp = \text{div}_x(\chi_n(e^{ia})^\perp) \to \text{div}_x(\chi(e^{ia})^\perp)$$

and at the limit

$$\nabla_x\chi \cdot (e^{ia})^\perp = -\partial_a \text{div } T^a u. \tag{III.6}$$

Then, since div $T^a u \in \mathcal{M}(\Omega \times \mathbb{R})$ as seen in Section II, we are in the same situation as in [JP] section 5.1, and we get the Sobolev regularity of φ by the theorem of DiPerna, Lions and Meyer [DLM] . □

IV Sign conditions for almost minimizing sequences

In this section, we consider sequences of u_n and φ_n satisfying the same hypotheses as previously and such that, in addition, $E_{\varepsilon_n}(u_n) \to 2|\partial\Omega|$.
We recall we know from [RS] that such sequences exist, and that we always have $\liminf E_{\varepsilon_n}(u_n) \geq 2|\partial\Omega|$ (see Theorem 1). Thus, the situation we consider corresponds in particular to minimizers of the energy, but not necessarily: we only assume convergence of the energy to the minimum of the limiting energy, but not that we have critical points or minimizers of E_{ε_n}.
We denote by $\mathcal{M}(\Omega)$ the space of bounded Radon measures on Ω.

Theorem 2 *Let $\varepsilon_n \to 0$ and $u_n \in H^1(\Omega, S^1)$ with a lifting $\varphi_n \in H^1(\Omega, \mathbb{R})$ i.e. $u_n = e^{i\varphi_n}$ a.e, and assume*

$$\begin{aligned} &\|\varphi_n\|_{L^\infty(\Omega)} \leq N \\ &E_{\varepsilon_n}(u_n) \to 2|\partial\Omega|. \end{aligned}$$

(Such a sequence exists, at least if $\partial\Omega$ is assumed to be analytic by parts). Writing $\mu_n = \nabla\varphi_n \cdot H_n$ and $\mu_n = \mu_n^+ - \mu_n^-$, where μ_n^+ and μ_n^- are the positive and negative parts of μ_n, then up to extraction, either

$$\text{(IV.1)} \qquad \begin{cases} \int_\Omega |\mu_n^-| \to 0 \\ \mu_n^+ \rightharpoonup \mu_{u,\varphi} \ \textit{weakly in } \mathcal{M}(\Omega) \\ \mu_{u,\varphi} = \text{div } (\varphi u + u^\perp) \geq 0 \end{cases}$$

or

$$\text{(IV.2)} \qquad \begin{cases} \int_\Omega |\mu_n^+| \to 0 \\ \mu_n^- \rightharpoonup \mu_{u,\varphi} \ \textit{weakly in } \mathcal{M}(\Omega) \\ \mu_{u,\varphi} = \text{div } (\varphi u + u^\perp) \leq 0. \end{cases}$$

(u, φ) is a minimizer of $\|\mu_{u,\varphi}\|$ over $\mathcal{C}$, i.e. $\|\mu_{u,\varphi}\| = |\partial\Omega|$; and writing $u = \nabla^\perp g$ with $g \in W_0^{1,\infty}(\Omega)$, we have $g \geq 0$ in the first case (respectively $g \leq 0$ in the second case).
Changing u_n to $-u_n$ if necessary, we can assume that (IV.1) holds and then

$$\forall a \in \mathbb{R} \qquad \text{div } T^a u \leq 0$$

i.e. div $T^a u$ *is a negative measure in Ω. In integral form:*

$$\text{(IV.3)} \qquad \forall f \in C^1(\mathbb{R}) \textit{ such that } f' \geq 0 \textit{ and } f \in L^1(\mathbb{R}^+), \quad \text{div } (F(\varphi), G(\varphi)) \geq 0,$$

where

$$F(t) = \int^t -f(s) \sin s \, ds \qquad G(t) = \int^t f(s) \cos s \, ds.$$

Proof: We recall that

$$E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\nabla\varphi_n \cdot H_n| \tag{IV.4}$$

and

$$\liminf_{n\to\infty} 2\int_\Omega |\nabla\varphi_n \cdot H_n| \geq 2\|\mu_{u,\varphi}\| \geq 2|\partial\Omega|.$$

Thus, if $E_{\varepsilon_n}(u_n) \to 2|\partial\Omega|$ we must have

$$\|\mu_{u,\varphi}\| = |\partial\Omega|. \tag{IV.5}$$

Lemma IV.1 *If $\|\mu_{u,\varphi}\| = |\partial\Omega|$ and g is defined by*

$$\begin{cases} u = \nabla^\perp g & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega, \end{cases}$$

then, either

$$\begin{cases} g \geq 0 & \text{in } \Omega \\ \mu_{u,\varphi} \geq 0 & \text{in } \Omega \end{cases}$$

or

$$\begin{cases} g \leq 0 & \text{in } \Omega \\ \mu_{u,\varphi} \leq 0 & \text{in } \Omega. \end{cases}$$

Proof: We follow exactly the proof of Lemma 5.1 of [RS]. Let f_ε be defined as follows:

$$\begin{cases} f_\varepsilon(s) = -1 & \text{if } s \leq -\varepsilon \\ f_\varepsilon(s) = 1 & \text{if } s \geq \varepsilon \\ f_\varepsilon(s) = \frac{s}{\varepsilon} & \text{if } -\varepsilon \leq s \leq \varepsilon. \end{cases} \tag{IV.6}$$

Let V be the level-set $\{g = 0\}$ and $V_\varepsilon = \{x \in \Omega, \mathrm{dist}(x, V) \leq \varepsilon\}$. On the one hand, since $|f_\varepsilon| \leq 1$,

$$\left|\int_\Omega f_\varepsilon(g)\mathrm{div}\ (\varphi u + u^\perp)\right| \leq \|\mu_{u,\varphi}\| = |\partial\Omega|. \tag{IV.7}$$

On the other hand, integrating by parts and using $g = 0$ on $\partial\Omega$,

$$\int_\Omega f_\varepsilon(g)\mathrm{div}\ (\varphi u + u^\perp) = -\int_\Omega f'_\varepsilon(g)\nabla g \cdot (\varphi u + u^\perp) = \frac{1}{\varepsilon}\int_{-\varepsilon\leq g\leq\varepsilon} |\nabla g|^2, \tag{IV.8}$$

where we have used the definition of f_ε and $u = \nabla^\perp g$. Then, since $|\nabla g| = 1$, this becomes

$$\int_\Omega f_\varepsilon(g)\mathrm{div}\ (\varphi u + u^\perp) = \frac{1}{\varepsilon}\mathrm{vol}(\{|g| \leq \varepsilon\}). \tag{IV.9}$$

Since $|\nabla g| = 1$, by the mean-value theorem, $|g| \le \varepsilon$ in V_ε, hence

$$\text{(IV.10)} \qquad \frac{1}{\varepsilon}\text{vol}(\{|g| \le \varepsilon\}) \ge \frac{1}{\varepsilon}\text{vol}(V_\varepsilon).$$

On the other hand,

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon}\text{vol}(V_\varepsilon) \ge \mathcal{H}^1(V).$$

Combining this with (IV.7)—(IV.10), we deduce that

$$\mathcal{H}^1(V) \le |\partial\Omega|.$$

But $g = 0$ on $\partial\Omega$ hence $\mathcal{H}^1(V) \ge |\partial\Omega|$, and there is equality. Consequently $\mathcal{H}^1(\{g = 0\} \cap \Omega) = 0$ and since g is Lipschitz, g cannot change sign in Ω hence $g \ge 0$ or $g \le 0$.

We suppose we are in the case $g \ge 0$, and we write $\mu_{u,\varphi} = \mu_{u,\varphi}^+ - \mu_{u,\varphi}^-$ where $\mu_{u,\varphi}^+$ and $\mu_{u,\varphi}^-$ are the positive and negative parts of $\mu_{u,\varphi}$. What precedes proves that

$$\int_\Omega f_\varepsilon(g)\mu_{u,\varphi} \ge |\partial\Omega| - o(1).$$

Hence,

$$\text{(IV.11)} \qquad |\partial\Omega| = \int_\Omega \mu_{u,\varphi}^+ + \mu_{u,\varphi}^- \ge \int_\Omega \mu_{u,\varphi}^+ \ge \int_\Omega f_\varepsilon(g)\mu_{u,\varphi}^+ \ge |\partial\Omega| + \int_\Omega f_\varepsilon(g)\mu_{u,\varphi}^- - o(1).$$

But, since $g \ge 0$, $f_\varepsilon(g) \ge 0$, and $\mu_{u,\varphi}^- \ge 0$, thus necessarily $\int_\Omega \mu_{u,\varphi}^+ = |\partial\Omega|$ and $\mu_{u,\varphi}^- = 0$. This means that $\mu_{u,\varphi} \ge 0$. The case $g \le 0$ is similar. □

Going back to (IV.4)

$$E_{\varepsilon_n}(u_n) \ge 2\int_\Omega |\nabla\varphi_n \cdot H_n| \ge 2\left|\int_\Omega \nabla\varphi_n \cdot H_n\right| \ge 2\left|\int_\Omega \mu_{u,\varphi}\right| - o(1) = 2|\partial\Omega| - o(1).$$

Thus, since $E_{\varepsilon_n}(u_n) \to 2|\partial\Omega|$, we must have

$$\int_\Omega |\nabla\varphi_n \cdot H_n| - \left|\int_\Omega \nabla\varphi_n \cdot H_n\right| = \int_\Omega \mu_n^+ + \mu_n^- - \left|\int_\Omega \mu_n^+ - \mu_n^-\right| \to 0.$$

Extracting a subsequence such that $\int_\Omega \mu_n^+ - \mu_n^-$ has a constant sign, we get that either $\int_\Omega \mu_n^- \to 0$ and $\mu_n^+ \rightharpoonup \mu_{u,\varphi} \ge 0$ or $\int_\Omega \mu_n^+ \to 0$ and $\mu_n^- \rightharpoonup \mu_{u,\varphi} \le 0$. This proves the first assertion of the theorem.

Then, we assume that we are in the first situation and we get back to (II.29), apply it to $f \ge 0$ and $\xi \ge 0$:

$$\text{(IV.12)} \qquad \int_\Omega (\mu_n^+ - \mu_n^-)f(\varphi_n)\xi(x) = -\int_{\Omega\times\mathbb{R}} f(a)\xi(x)\text{div}\,(T^a u_n + \chi_n H_n)\,dx\,da.$$

But,

$$\left|\int_\Omega \mu_n^- f(\varphi_n)\xi(x)\right| \le \|f\|_{L^\infty}\|\xi\|_{L^\infty}\int_\Omega \mu_n^- \to 0$$

and

$$\int_\Omega \mu_n^+ f(\varphi_n)\xi(x) \ge 0.$$

Therefore, passing to the limit in (IV.12) yields

$$\int_{\Omega\times\mathbb{R}} f(a)\xi(x)\text{div } T^a u \le 0.$$

This is true for all $f, \xi \ge 0$ in $C_0^\infty(\Omega)$, hence

$$\text{a.e. in } a \in \mathbb{R}, \quad \text{div } T^a u \le 0 \ \text{ in } \Omega,$$

and since $a \mapsto \text{div } T^a u$ is continuous, we can replace the a.e. by everywhere.

Next, we prove the integral form (IV.3). We multiply (III.5) by f and integrate over $\mathbb{R}$, we get

$$\text{(IV.13)} \qquad \text{div}_x \int_\mathbb{R} f(a)\chi(x,a)(e^{ia})^\perp = -\int_\mathbb{R} f(a)\partial_a(\text{div } T^a u)da.$$

We can integrate the right-hand side by parts. Observing that div $T^a u \equiv 0$ as soon as $a \ge \|\varphi\|_{L^\infty}$ or $a \le -\|\varphi\|_{L^\infty}$, there remains

$$\text{div}_x \int_\mathbb{R} f(a)\chi(x,a)(e^{ia})^\perp = \int_\mathbb{R} f'(a)\text{div } T^a u\, da.$$

The left-hand side is equal to

$$\text{div } \int_{\varphi(x)}^{+\infty} f(a)(-\sin a, \cos a)da,$$

and the integral converges because f was assumed to be in $L^1(\mathbb{R}^+)$. Its value is $(F(\infty) - F(\varphi(x)), G(\infty) - G(\varphi(x)))$, hence (IV.13) becomes

$$\text{div } (F(\varphi), G(\varphi)) = -\int_\mathbb{R} f'(a)\text{div } T^a u\, da \ge 0$$

using div $T^a u \le 0$ and $f' \ge 0$. This completes the proof of the theorem. □

Proof of the continuity of $a \mapsto \text{div } T^a u$*:*
The proof is inspired from [Pe]. Let $a_0 \in \mathbb{R}$ and h_ε be any family approaching the Heaviside function $h(a) = \mathbf{1}_{a \le a_0}$ in $L^1(\mathbb{R})$ as $\varepsilon \to 0$. Let us multiply (III.5) by $h_\varepsilon(a)$ and integrate, as in (IV.13) we obtain

$$\text{(IV.14)} \qquad -\text{div } (F_\varepsilon(\varphi), G_\varepsilon(\varphi)) = \int_\mathbb{R} h'_\varepsilon(a)\text{div } T^a u\, da,$$

where

$$\begin{cases} F_\varepsilon(t) = \displaystyle\int^t -h_\varepsilon(s)\sin s\, ds \\ G_\varepsilon(t) = \displaystyle\int^t h_\varepsilon(s)\cos s\, ds. \end{cases}$$

But since $h_\varepsilon \to h$ in $L^1(\mathbb{R})$, we have

$$F_\varepsilon(t) \to F(t) = \int^t -h(s)\sin s\, ds$$
$$G_\varepsilon(t) \to G(t) = \int^t h(s)\cos s\, ds,$$

and one can easily check that $(F(t), G(t)) = e^{iT^{a_0}t} + cste$. Therefore,

$$\text{div } (F_\varepsilon(\varphi(x)), G_\varepsilon(\varphi(x)) \underset{\varepsilon\to 0}{\longrightarrow} \text{div } T^{a_0}u \quad \text{in } \mathcal{D}'(\Omega).$$

Combining this with (IV.14),

$$\text{(IV.15)} \qquad -\int_{\mathbb{R}} h'_\varepsilon(a)\text{div } T^a u\, da \underset{\varepsilon\to 0}{\longrightarrow} \text{div } T^{a_0}u \quad \text{in } \mathcal{D}'(\Omega),$$

while $-h'_\varepsilon \to \delta_{a_0}$. This is true for any h_ε approaching h in L^1, hence (IV.15) yields the continuity of $a \mapsto \text{div } T^a u$ from $\mathbb{R}$ to $\mathcal{D}'(\Omega)$. □

Remark IV.1: Using the method of [Pe] on (III.5) with the sign condition on div $T^a u$, and this continuity result, one may establish that if (u_1, φ_1) and (u_2, φ_2) such that $\varphi_1 = \varphi_2$ on $\partial\Omega$ both satisfy (I.12) and (I.13), then $min(\varphi_1, \varphi_2)$ also does (hence is also a minimizer).

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