CS Peskin 5/22/2019

Mechanics of a network of rigid links with point masses at the vertices

We consider a collection of point masses

(1) $M_k, \quad k = 1 \ldots K$

<u>some</u> pairs of which are connected by rigid links.* Let

(2) $\underline{X}_k(t)$ = position of the k^{th} mass

(3) $\underline{U}_k(t)$ = velocity of the k^{th} mass

(4) $\underline{F}_k(t)$ = external force applied to the k^{th} mass

Let the rigid links be numbered $l = 1 \cdots L$, and let

(5) $k_1(l), k_2(l)$ = indices of the masses that are joined by link l

(6) R_l = (given) length of link l

(7) $T_l(t)$ = tension in link l

*The structure as a whole may or may not be rigid.

$$(8) \qquad \underline{E}_\ell = \frac{\underline{X}_{k_2(\ell)} - \underline{X}_{k_1(\ell)}}{R_\ell}$$

The equations of motion are, for $k = 1 \cdots K$,

$$(9) \qquad M_k \frac{d\underline{U}_k}{dt} = \underline{F}_k + \sum_{\ell'=1}^{L} T_{\ell'} \underline{E}_{\ell'} \left(\delta_{k k_1(\ell')} - \delta_{k k_2(\ell')} \right)$$

$$(10) \qquad \frac{d\underline{X}_k}{dt} = \underline{U}_k$$

Equations for the tensions are derived by differentiating the constraint equations twice with respect to time and making use of (10) and then (9).

For each link $\ell = 1 \cdots L$, we have

$$\| \underline{X}_{k_2(\ell)} - \underline{X}_{k_1(\ell)} \|^2 = R_\ell^2 \tag{11}$$

and therefore

$$\left(\underline{X}_{k_2(\ell)} - \underline{X}_{k_1(\ell)} \right) \cdot \left(\underline{U}_{k_2(\ell)} - \underline{U}_{k_1(\ell)} \right) = 0 \tag{12}$$

$$\left(\underline{X}_{k_2(\ell)} - \underline{X}_{k_1(\ell)} \right) \cdot \left(\frac{d\underline{U}_{k_2(\ell)}}{dt} - \frac{d\underline{U}_{k_1(\ell)}}{dt} \right) \tag{13}$$

$$= - \| \underline{U}_{k_2(\ell)} - \underline{U}_{k_1(\ell)} \|^2$$

From equation (9),

$$\frac{d\underline{U}_{k_2(\ell)}}{dt} - \frac{d\underline{U}_{k_1(\ell)}}{dt} = \tag{14}$$

$$\left(\frac{\underline{F}_{k_2(\ell)}}{M_{k_2(\ell)}} - \frac{\underline{F}_{k_1(\ell)}}{M_{k_1(\ell)}} \right)$$

$$+ \sum_{\ell'=1}^{L} T_{\ell'} \underline{E}_{\ell'} \left(\frac{\delta_{k_2(\ell)k_1(\ell')} - \delta_{k_2(\ell)k_2(\ell')}}{M_{k_2(\ell)}} \right.$$

$$\left. - \frac{\delta_{k_1(\ell)k_1(\ell')} - \delta_{k_1(\ell)k_2(\ell')}}{M_{k_1(\ell)}} \right)$$

Now we substitute (14) into (13) and multiply both sides by $-1/R_\ell$ to obtain the following

(15)
$$\underline{E}_\ell \cdot \sum_{\ell'=1}^{L} T_{\ell'} \underline{E}_{\ell'} \left(\left(\frac{\delta_{k_1(\ell)k_1(\ell')} - \delta_{k_1(\ell)k_2(\ell')}}{M_{k_1(\ell)}} \right) + \left(\frac{\delta_{k_2(\ell)k_2(\ell')} - \delta_{k_2(\ell)k_1(\ell')}}{M_{k_2(\ell)}} \right) \right)$$

$$= \frac{\| \underline{U}_{k_2(\ell)} - \underline{U}_{k_1(\ell)} \|^2}{R_\ell} + \underline{E}_\ell \cdot \left(\frac{\underline{F}_{k_2(\ell)}}{M_{k_2(\ell)}} - \frac{\underline{F}_{k_1(\ell)}}{M_{k_1(\ell)}} \right)$$

We now consider the possible values of the expression

$$(16) \qquad \frac{\delta_{k_1(l)k_1(l')} - \delta_{k_1(l)k_2(l')}}{M_{k_1(l)}} + \frac{\delta_{k_2(l)k_2(l')} - \delta_{k_2(l)k_1(l')}}{M_{k_2(l)}}$$

We assume that the two points joined by any particular link are different and that there is at most one link joining any given pair of points.

When $l = l'$, we obviously have

$$(17) \qquad \begin{cases} k_1(l) = k_1(l') & k_1(l) \neq k_2(l') \\ k_2(l) = k_2(l') & k_2(l) \neq k_1(l') \end{cases}$$

so (16) reduces to

$$(18) \qquad \frac{1}{M_{k_1(l)}} + \frac{1}{M_{k_2(l)}}$$

when $l' = l$.

When $\ell \neq \ell'$, the links ℓ and ℓ' have at most one end point in common, but there are four <u>mutually exclusive</u> ways that this can happen, namely

$$k_i(\ell) = k_j(\ell') \quad \text{for } i = 1,2 \text{ and } j = 1,2 \tag{19}$$

and then the expression (16) reduces to

$$\frac{(-1)^{i+j}}{M_{k_i(\ell)}} = \frac{(-1)^{i+j}}{M_{k_j(\ell')}} \tag{20}$$

Let

$$S(\ell,\ell') = \begin{cases} (-1)^{i+j} & \text{if } \ell \neq \ell' \text{ \underline{and} } (i,j) \text{ is such that } k_i(\ell) = k_j(\ell') \\ 0 & \text{if } \ell = \ell' \text{ \underline{or} } \nexists (i,j) \text{ such that } k_i(\ell) = k_j(\ell') \end{cases} \tag{21}$$

(For any particular ℓ, ℓ' such that $\ell \neq \ell'$, there is at most one pair (i,j) such that $k_i(\ell) = k_j(\ell')$.)

Also, if $S(l,l') \neq 0$, let

(22) $$M(l,l') = M_{k_i(l)} = M_{k_j(l')}$$

where (i,j) is such that $k_i(l) = k_j(l')$. (We regard $M(l,l')$ as undefined for l,l' such that $S(l,l')=0$.) Note that $M(l,l')$ is the mass at the vertex that is common to links l and l' (if there is such a vertex). Also

(23) $$S(l,l') = S(l',l)$$

(24) $$M(l,l') = M(l',l)$$

Equation (15) can now be rewritten as follows

(25)
$$\left(\frac{1}{M_{k_1(\ell)}} + \frac{1}{M_{k_2(\ell)}}\right) T_\ell$$

$$+ \sum_{\ell' : S(\ell,\ell') \neq 0} \left(\underline{E}_\ell \cdot \underline{E}_{\ell'}\right) \frac{S(\ell,\ell')}{M(\ell,\ell')} T_{\ell'} = B_\ell$$

where

(26)
$$B_\ell = \frac{\| \underline{U}_{k_2(\ell)} - \underline{U}_{k_1(\ell)} \|^2}{R_\ell}$$

$$+ \underline{E}_\ell \cdot \left(\frac{\underline{F}_{k_2(\ell)}}{M_{k_2(\ell)}} - \frac{\underline{F}_{k_1(\ell)}}{M_{k_1(\ell)}} \right)$$

The system (15), which is equivalent to (25) may be singular. To investigate this consider a solution $\tilde{T}_1 \cdots \tilde{T}_L$ of the corresponding homogeneous system:

$$(27) \quad \underline{E}_l \cdot \sum_{l'=1}^{L} \tilde{T}_{l'} \underline{E}_{l'} \left(\frac{\delta_{k_1(l) k_1(l')} - \delta_{k_1(l) k_2(l')}}{M_{k_1(l)}} + \frac{\delta_{k_2(l) k_2(l')} - \delta_{k_2(l) k_1(l')}}{M_{k_2(l)}} \right)$$

$$= 0$$

Now multiply both sides of (27) by $\tilde{T}_l$ and sum over l. This gives

(28)
$$\sum_{l,l'=1}^{L} \tilde{T}_l \tilde{T}_{l'} \underline{E}_l \cdot \underline{E}_{l'}$$

$$\left(\frac{\delta_{k_1(l)k_1(l')} - \delta_{k_1(l)k_2(l')}}{M_{k_1(l)}} + \frac{\delta_{k_2(l)k_2(l')} - \delta_{k_2(l)k_1(l')}}{M_{k_2(l)}} \right)$$

$$= 0$$

But for $i = 1,2$, we have

(29)
$$\frac{\delta_{k_i(l)k_1(l')} - \delta_{k_i(l)k_2(l')}}{M_{k_i(l)}}$$

$$= \sum_{k=1}^{K} \frac{1}{M_k} \delta_{k k_i(l)} \left(\delta_{k k_1(l')} - \delta_{k k_2(l')} \right)$$

Making use of (29), we can rewrite (28) as follows:

(30) $$0=\sum_{k=1}^{K}\frac{1}{m_k}\sum_{l,l'=1}^{L}\tilde{T}_l\tilde{T}_{l'}\,\underline{E}_l\cdot\underline{E}_{l'}$$

$$\left(\delta_{kk_1(l)}-\delta_{kk_2(l)}\right)\left(\delta_{kk_1(l')}-\delta_{kk_2(l')}\right)$$

$$=\sum_{k=1}^{K}\frac{1}{m_k}\left\|\sum_{l'=1}^{L}\tilde{T}_{l'}\underline{E}_{l'}\left(\delta_{kk_1(l')}-\delta_{kk_2(l')}\right)\right\|^2$$

and of course it follows from this that

(31) $$\sum_{l'=1}^{L}\tilde{T}_{l'}\underline{E}_{l'}\left(\delta_{kk_1(l')}-\delta_{kk_2(l')}\right)=0$$

for $k=1\cdots K$. Note that the left-hand side of (31) has exactly the form of the force that is applied by the tensions in the rigid links to the mass at vertex k, see equation (9).

If the system (15), or equivalently (25), has more than one solution, then the difference between any two solutions satisfies (27), and therefore also (31), which is derived from (27). This shows that the non-uniqueness is harmless, since the difference of the tensions does not lead to any difference at all in the forces that are applied to the masses of the system.

We still have to prove existence of solutions to the system (15) $\Leftrightarrow$ (25). Since this system is symmetric, as is clear from (25), we have to show that any $\tilde{T}_1 \cdots \tilde{T}_L$ that satisfies (27)

is orthogonal to the right-hand side, i.e., that

$$\sum_{l=1}^{L} B_l \tilde{T}_l = 0 \qquad (32)$$

for all $\tilde{T}_1 \cdots \tilde{T}_L$ that satisfy (27), and therefore (31).

We consider the two terms of B_ℓ separately, see (26). We have

$$\sum_{\ell=1}^{L} \tilde{T}_\ell \underline{E}_\ell \cdot \left(\frac{\underline{F}_{k_2(\ell)}}{M_{k_2(\ell)}} - \frac{\underline{F}_{k_1(\ell)}}{M_{k_1(\ell)}} \right) \tag{33}$$

$$= \sum_{k=1}^{K} \sum_{\ell=1}^{L} \tilde{T}_\ell \underline{E}_\ell \left(\delta_{k k_2(\ell)} - \delta_{k k_1(\ell)} \right) \cdot \frac{\underline{F}_k}{M_k}$$

$$= 0$$

as required, see (31).

To complete the proof of existence, we need to show that

$$(34) \qquad \sum_{\ell=1}^{L} \tilde{T}_\ell \, \frac{\| \underline{U}_{k_2(\ell)} - \underline{U}_{k_1(\ell)} \|^2}{R_\ell} = 0$$

for all $(\tilde{T}_1 \cdots \tilde{T}_L)$ that satisfy (27) and therefore (31). This will not be true for arbitrary $(\underline{U}_1 \cdots \underline{U}_K)$, and we require an additional hypothesis: that $(\underline{X}_1 \cdots \underline{X}_K)$ and $(\underline{U}_1 \cdots \underline{U}_K)$ are such that there _exist_ accelerations $(\underline{A}_1 \cdots \underline{A}_K)$ that are consistent with the constraints in the following sense:

$$(35) \qquad (\underline{X}_{k_2(l)} - \underline{X}_{k_1(l)}) \cdot (\underline{A}_{k_2(l)} - \underline{A}_{k_1(l)})$$

$$= - \| \underline{U}_{k_2(l)} - \underline{U}_{k_1(l)} \|^2$$

for $l = 1 \cdots L$, see equation (13), but note that the $\underline{A}_k$ in (35) need not be the same as the physical accelerations $d\underline{U}_k / dt$. Equation (35) is a purely kinematic condition. In particular, it does not involve the masses M_k.

Note that we require only the existence of $(\underline{A}_1 \cdots \underline{A}_K)$ that satisfy (35), and that this, in turn, is a restriction on $(\underline{X}_1 \cdots \underline{X}_K, \underline{U}_1 \cdots \underline{U}_K)$.

From the hypothesis (35), it is straightforward to prove (34). Multiplying both sides of (35) by $\tilde{T}_\ell / R_\ell$ and summing over ℓ, we get

(36)
$$-\sum_{\ell=1}^{L} \tilde{T}_\ell \frac{\|\underline{U}_{k_2(\ell)} - \underline{U}_{k_1(\ell)}\|^2}{R_\ell}$$

$$= \sum_{\ell=1}^{L} \tilde{T}_\ell \underline{E}_\ell \cdot \left(\underline{A}_{k_2(\ell)} - \underline{A}_{k_1(\ell)}\right)$$

$$= \sum_{k=1}^{K} \sum_{\ell=1}^{L} \tilde{T}_\ell \underline{E}_\ell \left(\delta_{k k_2(\ell)} - \delta_{k k_1(\ell)}\right) \cdot \underline{A}_k$$

$$= 0,$$

see equation (31). This completes the proof of existence of the tensions $T_1 \cdots T_L$.

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Thanks to Miranda Holmes-Cerfon for pointing out the need to make the hypothesis (35) instead of equation (12) in order to prove existence of the tensions, since there are special cases in which velocities that satisfy (12) are nevertheless inconsistent with the constraints, since they imply the non-existence of accelerations that satisfy (35).