HW12 - Due 04/30/2008  
ODE - spring 2008

This HW will count as 1/3 of the final grade.

1) Study the stability of the critical points of \( x' = 1 - 2\mu x + x^2 \) where \( \mu \) is a parameter.

2) Consider \( x'' + \sin(x) = 0 \) with \( x(0) = a \) and \( x'(0) = 0 \) and \( 0 < a < \pi \).
   a/ Prove that the solution is periodic in time with period \( T(a) \).
   b/ Find an expansion of \( T(a) \) when \( a \) goes to 0 (just compute the first two terms).
   c/ What is the behaviour of \( T(a) \) when \( a \) goes to \( \pi \).
   d/ What happens for other values of \( a \in \mathbb{R} \).

3) Consider the following system with one prey and two predators

\[
\begin{align*}
x' &= ax - xy - xz \\
y' &= -by + xy \\
z' &= -cz + xz
\end{align*}
\]

with \( a, b, c > 0 \). Take an initial data \( x(0), y(0), z(0) \geq 0 \).
   a/ Prove that the system has a global solution and that for \( t \geq 0 \), we have \( x(t), y(t), z(t) \geq 0 \).
   b/ What are the equilibrium solutions. Characterise them.
   c/ Are there periodic solutions with \( y(t)z(t) > 0 \)?
   d/ Is it possible that starting from \( y(0), z(0) > 0 \), one of the predators becomes extinct in finite time? or in infinite time?

PS: Please check for updated versions if there are any corrections.