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Averaging over Fast Gravity Waves for Geophysical Flows with Unbalanced Initial Data[1]

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Abstract. Various facets of recent mathematical theories for averaging over fast gravity waves on advective time scales for geophysical flows with unbalanced initial data are presented here including nonlinear Rossby adjustment and simplified reduced dynamics. This work is presented within the context of simplified geophysical models involving the rotating shallow-water equations and the rotating stably stratified Boussinesq equations. Novel mechanisms for enhanced gravity wave dissipation through the catalytic interaction with potential vortical modes are also developed here within the context of the rotating shallow-water equations.

1. Introduction

Here we discuss recent mathematically rigorous theories (Embid and Majda, 1996, 1997) for averaging over fast gravity waves for geophysical flows with unbalanced initial data. These theories are asymptotic in situations where the gravity waves have much higher frequencies than the potential vortical parts of the flow, i.e., the averaging theories apply on order one advective time scales in the low Froude number asymptotic limit with either a low Rossby number limit or at fixed Rossby numbers. We discuss the general mathematical framework for averaging over fast gravity-waves in two prototype models for geophysical flows: the rotating shallow water equations (RSWE) and the rotating stably stratified Boussinesq equations (RSSBE). We also briefly discuss prototype examples of averaging theories over the fast gravity waves for longer times, i.e., many large-scale eddy turnover times (Callet, 1997; Callet and Majda, 1997), and present prototype examples of enhanced gravity wave dissipation through catalytic interaction with the potential vortical modes over these longer time scales.

What is the motivation for studying the behavior of solutions of the equations such as RSWE or RSSBE with unbalanced initial data? There are two different physical motivations:

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1. Does nonlinear Rossby adjustment (Gill, 1982) occur in a suitable sense even for unbalanced initial data? In other words, even for unbalanced initial data, do the potential vortical modes move independently of the noisy sea of gravity modes and still continue to satisfy the quasigeostrophic equations in some averaged sense? (1)
2. Modelling of flows on the mesoscales in both the atmosphere and ocean naturally involves general unbalanced mixtures of vortical modes and gravity modes and their interaction. For prototype model equations such as the RSSBE, are there simplified dynamical equations describing this interaction at low Froude numbers with fixed or low Rossby numbers? (2)

Regarding (1), statistical theory (Warn, 1986) and numerical simulations (Farge and Sadourny, 1989; Bartello, 1995) in periodic geometry strongly suggest such nonlinear Rossby adjustment as described in (1). The recent mathematical theories (Embid and Majda, 1996, 1997) rigorously confirm this possibility that *nonlinear Rossby adjustment always occurs in a suitable averaged sense for general unbalanced initial data on order one advective time scales for either RSWE or RSBBE in the low Froude and low Rossby number limits with spatially periodic initial data*. We remark here that utilizing spatially periodic initial data is a severe test since the gravity waves cannot scatter and radiate away to infinity. One of the mathematical proofs of general nonlinear Rossby adjustment involves the systematic use of Ertel's theorem on conservation of potential vorticity (see Embid and Majda, 1996, 1997) while the other involves direct calculation.

Next, we provide more detailed discussion for the motivation in (2) for studying the RSSBE with unbalanced initial data as well as a context for the work in Embid and Majda (1997). The universal features of the velocity spectrum on mesoscales in the lower atmosphere (Gage, 1979; Gage and Nastrom, 1986) as well as the universal wave spectrum on somewhat smaller scales in the ocean (Garrett and Munk, 1979) have inspired a large theoretical effort attempting to explain these phenomena. Analytical efforts have focused on the three-wave resonant nonlinear interactions among internal gravity waves (McComas and Bretherton, 1977; Muller *et al.*, 1986) and more recently on three wave interactions for an individual triad with two gravity waves and a vortical mode (Lelong, 1989; Lelong and Riley, 1991). Following the pioneering work of Riley *et al.* (1981), recent large scale numerical simulations in idealized periodic geometry have studied solutions of the rotating Boussinesq equations with either continual forcing (Herring and Metais, 1989; Ramsden and Holloway, 1992; Metais *et al.* 1994) or free decay (Metais and Herring, 1989; Bartello, 1995) at a variety of Froude and Rossby numbers in an attempt to explain the universal observed spectra in an idealized setting. These numerical experiments reveal remarkable differences in the behavior of strongly stratified flows with low Froude number and fixed Rossby number compared with such flows at both low Froude number and low Rossby number (Bartello, 1997).

The purpose of the paper by Embid and Majda (1997) is to develop in detail the reduced limiting dynamics for solutions of the rotating Boussinesq equations both at low Froude number and fixed Rossby number as well as for the low Froude number and low Rossby number limit and then to compare and contrast these dynamics to provide a general analytical framework for explaining some of the remarkable features observed in the numerical simulations. To achieve this, the authors apply the recent mathematically rigorous theory for averaging over fast waves in geophysical flows (Embid and Majda, 1996). This theory involves a modification to account for the strong dispersion in geophysical flows of techniques developed earlier in the mathematical study of incompressible limits for compressible flows (Klainerman and Majda, 1981; Majda, 1984; Schochet, 1994) as well as judicious use of Ertel's theorem on conservation of potential vorticity. The general theory yields reduced limiting dynamics which include resonant triad interactions for the slow (vortical) modes, the effect of the slow (vortical) modes on the fast internal (gravity) modes, and also the general resonant triad interactions among internal gravity waves from earlier work (McComas and Bretherton, 1977).

We briefly summarize some crucial differences in the limiting dynamics. The reduced slow dynamics for the vortical modes in the low Froude number limit at fixed Rossby numbers includes vertically sheared horizontal motion while the reduced slow dynamics in the low Froude number and low Rossby number limit yields the familiar quasi-geostropic equations where such vertically sheared horizontal motion is completely absent—in fact, such vertically sheared motions participate only in the fast dynamics through resonant interactions in this quasi-geostrophic limit (also see Section 2 below). The reduced dynamics also explains the conservation in time of the energy ratio between vortical modes and gravity modes observed in decaying numerical simulations at low Froude numbers (Metais and Herring, 1989). Furthermore, the reduced equations derived for the slow–fast vortical and gravity wave interactions include those derived earlier by Lelong (1989), Lelong and Riley (1991), and Bartello (1995) as special cases.

Next we summarize the remaining content of this paper. We end the Introduction with a discussion of fast wave averaging in the simplest context involving the RSWE. In Section 2 we describe the set-up for fast wave averaging for RSSBE and qualitatively describe important differences between the low Froude number and fixed Rossby number and the low Froude and low Rossby number limiting dynamics. In Section 3 we describe some new results on long-time averaging of RSWE (Callet, 1997; Callet and Majda, 1997). Finally in Section 4 we present a short list of important problems for future investigation.

1.1. Fast Wave Averaging for the Rotating Shallow-Water Equations

We consider the nondimensionalized RSWE,

$$\begin{aligned} \frac{D\vec{v}}{Dt} + (Ro)^{-1}\vec{v}^{\perp} + (Fr)^{-2}\theta\nabla h &= 0, \\ \frac{Dh}{Dt} + \theta^{-1}\,\mathrm{div}\,\vec{v} + h\,\mathrm{div}\,\vec{v} &= 0, \end{aligned} \tag{3}$$

with $\vec{v} = (v_1, v_2), \vec{v}^{\perp} = (-v_2, v_1), \vec{x} = (x, y)$, and $D/Dt = \partial/\partial t + \vec{v}\cdot\nabla$, the convective derivative. The quantity h in (3) is the nondimensional height perturbation. For these equations the conservation of potential vorticity (Ertel's theorem) yields

$$\frac{D}{Dt}\left(\frac{1 + Ro\,\omega}{1 + \theta h}\right) = 0, \tag{4}$$

where $\omega = \partial v_2/\partial x - \partial v_1/\partial y$ is the relative vorticity. The nondimensional parameters in (3) are given by

$$\begin{aligned} Ro &= \frac{U}{Lf} && \text{(Rossby number)}, \\ Fr &= \frac{U}{\sqrt{gH_0}} && \text{(Froude number)}, \\ \theta &= \frac{N_0}{H_0} && \text{(height ratio)}. \end{aligned} \tag{5}$$

The units of time in (3) are the large-scale advection time, U/L, where L is the length scale. Since U/L defines the large-scale advection time while $1/f$ defines the rotation time, the Rossby number in (5) measures the ratio of the rotation time to the eddy turnover time. The Froude number, Fr, is the analogue of the Mach number in compressible flows and measures the ratio of the typical fluid velocity to the gravity wave speed. The height ratio θ represents the nondimensional ratio of the perturbation height, N_0, measuring the size of h to the mean height, H_0, of the undisturbed fluid. We hope that our use of θ in the above discussion and throughout the remainder of this subsection will create no confusion for the reader with other physically based quantities such as the potential temperature.

Low Froude Number Limiting Dynamics

Low Froude number limiting dynamics at a fixed Rossby number for RSWE in (3) is the analogue of the incompressible limit for compressible fluids. In this limit, we assume that

$$Fr = \varepsilon \ll 1, \qquad \theta = \varepsilon, \qquad Ro = O(1). \tag{6}$$

The analogy with the compressible and incompressible limit is perfect here since the equations in (3) describe an isentropic rotating compressible flow with gas constant, $\gamma = 2$, and the Froude number, Fr, in (5) is exactly the Mach number of the corresponding flow field.

Klainerman and Majda (1981, 1982) developed a rigorous mathematical theory for taking the incompressible limit of the RSWE in (3) under the distinguished limit is (6) provided that the initial data is nearly incompressible, i.e.,

$$\operatorname{div} \vec{v}_0 = O(\varepsilon), \qquad h_0 = O(\varepsilon). \tag{7}$$

Independently, Kreiss (1980) studied the same formal limiting procedure for RSWE from (6) but he required many more constraints on the initial data besides (7); namely that the solution of (3) formally has several time derivatives which are bounded at time $t = 0$ independent of ε. This procedure is the bounded time derivative initialization procedure (Browning *et al.*, 1980) which is equivalent to the Baer–Tribbia (1977) and Machenhauer (1977) initialization schemes for filtering gravity waves. Klainerman and Majda (1982) have established that the bounded time derivative method of Kreiss (1980) is a special case of their more general mathematical procedure. A simplified treatment of the compressible and incompressible limit is presented in Chapter 2 of Majda's monograph (1984) and is the starting place which the authors recommend for the reader interested in learning the mathematical theory described briefly in this paper. The authors' forthcoming monograph also contains these ideas in various geophysical contexts (Majda and Embid, 1995).

What happens to the low Froude number limiting behavior of RSWE as $\varepsilon \downarrow 0$ on advective time scales with general unbalanced initial data so that (7) is no longer satisfied? This question was posed by one of the authors (Majda, 1984) and answered by Schochet (1994) in a recent important paper. The idea of the mathematical argument is to utilize a new variable which "cancels the oscillations" to leading order and then to apply the arguments from Chapter 2 of Majda (1984) straightforwardly to this new variable. Embid and Majda (1996) have explained Schochet's "cancellation of oscillations" technique through a simple asymptotic procedure involving the method of averaging for partial differential equations. Independently of Schochet, Grenier (1995) also has utilized the cancellation of oscillations device in recent work regarding fluid flow. This completes our historical discussion of the low Froude number limit of RSWE. Next, we present a more detailed discussion of the

Low Froude Number, Low Rossby Number Limit for the Rotating Shallow-Water Equations

We consider the RSWE with the familiar quasi-geostrophic scaling (Pedlosky, 1987)

$$Ro = \varepsilon, \quad Fr = F^{1/2}\varepsilon, \quad \theta = F\varepsilon, \tag{8}$$

with fixed $F > 0$ and $\varepsilon \ll 1$. By utilizing (8) in (3) together with the rescaling, $h_{old} = F^{-1/2}h_{\text{new}}$, we obtain the rapidly RSWE

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + \varepsilon^{-1}\vec{v}^{\perp} + \varepsilon^{-1}F^{-1/2}\nabla h + \vec{v}\cdot\nabla\vec{v} &= 0, \\ \frac{\partial h}{\partial t} + \varepsilon^{-1}F^{-1/2} \operatorname{div} \vec{v} + \vec{v}\cdot\nabla h + h \operatorname{div} \vec{v} &= 0. \end{aligned} \tag{9}$$

The equations in (9) have the general

Abstract Form of Rapidly Rotating Shallow-Water Equations

$$\frac{\partial \vec{u}}{\partial t} + \varepsilon^{-1}\mathcal{L}(\vec{u}) + \mathcal{B}(\vec{u},\vec{u}) = 0, \tag{10}$$

where

$$\begin{aligned} \vec{u} &= (\vec{v}, h), \\ \mathcal{L}(\vec{u}) &= \begin{pmatrix} \vec{v}^{\perp} + F^{-1/2}\nabla h \\ F^{-1/2} \operatorname{div} \vec{v} \end{pmatrix}, \\ \mathcal{B}(\vec{u},\vec{u}) &= \begin{pmatrix} \vec{v}\cdot\nabla\vec{v} \\ \operatorname{div}(h\vec{v}) \end{pmatrix}. \end{aligned} \tag{11}$$

The linear operator, $\mathcal{L}(\vec{u})$, in (10) is skew symmetric and has three imaginary eigenvalues for any wave number $\vec{k} = (k_1, k_2)$ given by $i\omega(\vec{k})$ where

$$\begin{aligned} \omega^0(\vec{k}) &= 0 \qquad \text{(the potential vortical mode)}, \\ \omega^{\pm}(\vec{k}) &= \pm(1 + F^{-1}|\vec{k}|^2)^{1/2} \qquad \text{(the inertio-gravity modes)}. \end{aligned} \tag{12}$$

Thus, looking back at (10), we see that the rapidly RSWE have slow mode propagation on advective time scales given by the potential vortical mode and fast modes of propagation given by the inertio-gravity modes.

A standard formal power series expansion of (10) and Ertel's potential vorticity theorem (Pedlosky, 1987; Majda and Embid, 1995) yields the familiar formal leading order behavior given by the

Quasi-Geostrophic Equations

$$\begin{aligned} &\vec{v} = \nabla^{\perp} h, \\ &q = \Delta h - F h, \\ &\frac{Dq}{Dt} = 0. \end{aligned} \tag{13}$$

A rigorous mathematical justification of this limiting procedure for initial data in geostrophic balance is given by Schochet (1987) following the basic strategy of Klainerman and Majda (1981, 1982) discussed earlier.

What happens to solutions of rapidly RSWE as $\varepsilon \to 0$ for general unbalanced initial data? The abstract form in (10) is useful since it suggests that we apply a version of the method of averaging, perhaps familiar to the reader for ordinary differential equations (Kevorkian and Cole, 1978; Jordan and Smith, 1976). Under the assumption that the linear operator, $\mathcal{L}(\vec{u})$, in (10) is skew symmetric so that $e^{-\mathcal{L}\tau}$ has only oscillatory modes, we seek formal asymptotic solutions of (10) with two time scales:

$$\vec{u}^{\varepsilon}(\vec{x}, t) = \vec{u}^0(\vec{x}, t, \tau)|_{\tau=t/\varepsilon} + \varepsilon \vec{u}^1(\vec{x}, t, \tau)|_{\tau=t/\varepsilon} + O(\varepsilon^2). \tag{14}$$

Careful application of the method of multiple time scales (see Embid and Majda, 1996, 1997) yields the leading-order behavior

$$\vec{u}^0(\vec{x}, t, \tau) = e^{-\mathcal{L}\tau} \overline{u}(\vec{x}, t), \tag{15}$$

where $\overline{u}(\vec{x}, t)$ satisfies the

Averaged Equation

$$\begin{aligned} \frac{\partial \overline{u}}{\partial t}(\vec{x}, t) + \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} e^{s\mathcal{L}} \mathcal{B}(e^{-s\mathcal{L}} \overline{u}(\vec{x}, t), e^{-s\mathcal{L}} \overline{u}(\vec{x}, t)) ds = 0, \\ \overline{u}(\vec{x}, t)|_{t=0} = \vec{u}^0(\vec{x}) \end{aligned} \tag{16}$$

with $\vec{u}^0(\vec{x})$ being the general unbalanced initial data for (10). For spatially periodic initial data, the linear operator, $\mathcal{L}$, has the eigenvalue decomposition

$$e^{-\tau\mathcal{L}} f = \sum_{\alpha=0,\pm1} \sum_{\vec{k}} e^{i(\vec{k}\cdot\vec{x} - \omega_\alpha(\vec{k})\tau)} \sigma_{\vec{k}}^{(\alpha)} \vec{r}_{\vec{k}}^{(\alpha)}, \tag{17}$$

where $\vec{r}_{\vec{k}}^{(\alpha)}$ are the eigenvectors of the linear problem and $\sigma_{\vec{k}}^{(\alpha)}$ are the basis expansion coefficients of the function f. Inserting (17) into (16), it is easy to compute that if we expand $\overline{u}(\vec{x}, t)$ as

$$\overline{u}(\vec{x}, t) = \sum_{\alpha=0,\pm1} \sigma_{\vec{k}}^{(\alpha)}(t) \vec{r}_{\vec{k}}^{(\alpha)}, \tag{18}$$

then the only contributions to the time rate of change of $\sigma_{\vec{k}}^{(\alpha)}(t)$ arise from the *direct three wave resonances*

$$\begin{aligned} \vec{k} &= \vec{k}_1 + \vec{k}_2, \\ \omega^{(\alpha)}(\vec{k}) &= \omega^{(\alpha_1)}(\vec{k}_1) + \omega^{(\alpha_2)}(\vec{k}_2). \end{aligned} \tag{19}$$

The explicit form for these reduced limiting dynamic equations for the RSWE in (9) are found in Embid and Majda (1996).

Obviously, the method which we have just outlined in (10)–(19) to derive the reduced limiting dynamics on advective time scales for rapidly RSWE is a general systematic procedure and applies to many other equations besides the singular limit of rapidly RSWE. In Section 2 we review how to achieve a similar framework for the various singular limits of RSSBE and the explicit form of the limiting dynamics for these cases can be found in Embid and Majda (1997).

With the formal procedure for computing limiting dynamics which we have just presented, it is easy to motivate Schochet's "cancellation of oscillations" trick (Embid and Majda, 1996). Define $\bar{u}(\vec{x}, t)$ by

$$\bar{u}_\varepsilon(\vec{x}, t, \tau) = e^{\mathcal{L}(t/\varepsilon-\tau)}\vec{u}^\varepsilon(\vec{x}, t),$$

where by construction $\bar{u}_\varepsilon(\vec{x}, t, \tau)$ cancels enough of the oscillations so that its time derivative stays bounded; then apply the standard generalized theory of Klainerman and Majda (1981, 1982) and Majda (1984) to $\bar{u}_\varepsilon(\vec{x}, t, \tau)$; the technical details are found in Schochet (1994).

Thus, the rigorous theory guarantees that, as $\varepsilon \to 0$, the solutions of the rapidly RSWE have the asymptotic structure

$$\vec{u}^\varepsilon(\vec{x}, t) = \bar{u}_{\mathrm{I}}(\vec{x}, t) + e^{-t/\varepsilon\mathcal{L}}\bar{u}_{\mathrm{II}}(\vec{x}, t) + o(1), \tag{20}$$

where the component, $\vec{u}_{\mathrm{I}}(\vec{x}, t)$, is determined by the quasi-geostrophic equations in (13) with initial data determined by the projection of the unbalanced initial data on the component in geostrophic balance, i.e., $\vec{u}_{\mathrm{I}} = (\vec{v}_{\mathrm{I}}, h_{\mathrm{I}})$ satisfies $\vec{v}_{\mathrm{I}} = \nabla^\perp h_{\mathrm{I}}$. The gravity wave component $\vec{u}_{\mathrm{II}}(\vec{x}, t)$ is given by Fourier modes in energy shells:

$$\bar{u}_{\mathrm{II}}(\vec{x}, t) = \sum_{l=1}^{\infty} \sum_{|\vec{k}|=\Lambda_l} e^{i\vec{k}\cdot\vec{x}} \left(\sum_{\alpha=\pm1} \sigma^{(\alpha)}_{(\vec{k})} \vec{r}^{(\alpha)}_{(\vec{k})} \right), \tag{21}$$

where the dynamics of $\vec{u}_{\mathrm{II}}(\vec{x}, t)$ are given at each energy level $|\vec{k}| = \Lambda_l$ by the reduced dynamics equations

$$\frac{\partial \sigma^{(\alpha)}_{(\vec{k})}}{\partial t} = \sum_{|\vec{k}_1|=\Lambda_l} \mathcal{C}^{(\alpha)}_{(\vec{k}_1,\vec{k})}(\vec{u}_{\mathrm{I}})\sigma^{(\alpha)}_{(\vec{k}_1)}, \qquad \alpha = \pm1, \tag{22}$$

where $(\mathcal{C}^{(\alpha)}_{(\vec{k}_1,\vec{k})}(\vec{u}_{\mathrm{I}}))$ is a skew Hermitian matrix depending on the quasi-geostrophic component $\vec{u}_{\mathrm{I}}$. The formulas in (21) and (22) provide a rigorous proof for the limiting dynamics conjectured by Warn (1986) from statistical theories. From (21), we calculate explicitly that

$$e^{-(t/\varepsilon)\mathcal{L}}\bar{u}_{\mathrm{II}}(\vec{x}, t) = \sum_{l=1}^{\infty} \sum_{|\vec{k}|=\Lambda_l} e^{i(\vec{k}\cdot\vec{x}-(t/\varepsilon)\omega^\alpha(\vec{k}))}\sigma^{(\alpha)}_{\vec{k}}\vec{r}^{(\alpha)}_{\vec{k}}, \tag{23}$$

with $\alpha = \pm1$.

Nonlinear Rossby Adjustment

Finally, we explain the fashion in which nonlinear Rossby adjustment occurs even for unbalanced initial data for rapidly RSWE. First, the results which we have just described in (20)–(23) show that we could first initialize the problem by projecting the unbalanced initial data on the set of initial data in geostrophic balance, i.e., satisfying $\vec{v}_0 = \nabla^\perp h_0$, and solve the initial value problem or alternatively we could solve the initial value problem with general unbalanced initial data and later project this noisy solution on the components in geostrophic balance. The mathematical theory which we have just described guarantees rigorously that these two procedures agree at least for $\varepsilon \ll 1$ and a fixed interval on the advective time scale for any unbalanced initial data. Both M. Cullen and O. Talagrand independently have communicated to the authors that exactly such effects are often observed in 1–2 day short-range forecasts.

Furthermore, with suitable space-time filtering, nonlinear Rossby adjustment occurs as $\varepsilon \to 0$ because the fast gravity wave contributions average to zero. We consider space-time filtering by multiplying $\vec{u}^\varepsilon(\vec{x}, t)$

by any smooth filter function $\vec{\phi}(\vec{x}, t)$, i.e., we consider

$$\int\int \vec{\phi}(\vec{x}, t) \cdot \vec{u}^{\varepsilon}(\vec{x}, t)\, d\vec{x}\, dt. \tag{24}$$

Then, it follows from (20), (23), and (24) that, for $\varepsilon \ll 1$,

$$\int\int \vec{\phi}(\vec{x}, t) \cdot \vec{u}^{\varepsilon}(\vec{x}, t)\, d\vec{x}\, dt = \int\int \vec{\phi}(\vec{x}, t) \cdot \overline{u}_{\mathrm{I}}(\vec{x}, t)\, d\vec{x}\, dt + o(1). \tag{25}$$

Thus, only the quasi-geostrophic component of the solution $\overline{u}_{\mathrm{I}}$ is observed to persist through a space-time filter for $\varepsilon \ll 1$ and the leading-order contributions to (25) from the inertio-gravity waves in (23) always average to zero as ε tends to zero. This is the general nonlinear Rossby adjustment process mentioned earlier.

Also, the exact solution described in (21), (22) for the reduced inertio-gravity wave dynamics shows that, as $\varepsilon \to 0$, the cascade of gravity wave energy is completely suppressed to leading order on advective time scales for the rapidly RSWE without any dissipation. Even when dissipation is incorporated (see Section 2 and Embid and Majda (1997) for RSSBE) the theory yields that the long wavelength gravity modes necessarily remain out of balance for $\varepsilon \ll 1$ and geostrophic balance is never achieved when the solution is sampled at a fixed time without space-time filtering. Thus, the theory explains the results of numerical simulations by Farge and Sadourny (1989) where balance was never observed for $\varepsilon \ll 1$ and unbalanced initial data. A more systematic explanation of nonlinear Rossby adjustment with general unbalanced initial data can be developed through the use of Ertel's potential vorticity equation in (4) coupled with the fact that to leading order in ε, the gravity waves do not contribute to the potential vorticity. Lack of space prevents detailed development of these arguments here although they are presented in detail in Embid and Majda (1997) for the more general limiting dynamics for RSSBE with general slanted rotation.

2. Fast Wave Averaging for the Rotating Stably Stratified Boussinesq Equations

Here we describe the fast wave averaging theories for the RSSBE and the limiting dynamics which emerge respectively in the low Froude number and fixed Rossby number and low Froude, low Rossby number limits on advective time scales. In both cases we show how to write the RSSBE in the abstract form in (10) so that the abstract averaged equation from (16) can be utilized to compute the limiting dynamics. We also briefly compare and contrast the limiting dynamics which emerge in these two situations. The interested reader can consult Embid and Majda (1997) for a much more detailed comparison as well as for the elaborate algebraic manipulation which is omitted here.

2.1. The Rotating Stably Stratified Boussinesq Equations

The RSSBE including dissipative effects are given by

$$\begin{aligned} \frac{D\vec{v}}{Dt} + f\vec{\eta} \times \vec{v} + \rho_b^{-1}\rho\nabla\phi + \rho_b^{-1}\rho\vec{e}_3 &= \mu\rho_b^{-1}\Delta\vec{v}, \\ \frac{D\rho}{Dt} - bw &= D\Delta\rho, \\ \operatorname{div} \vec{v} &= 0, \end{aligned} \tag{26}$$

with

$$\begin{aligned} &\vec{x} = (x, y, z), \\ &\vec{\eta} = (0, 0, 1), \\ &\left.\begin{aligned} &\tilde{\rho} = \overline{\rho} + \rho \\ &\overline{\rho} = \rho_b - bz, \quad b > 0 \end{aligned}\right\} \qquad \text{(Stable stratification)}, \\ &\vec{v} = (v_1, v_2, w) \qquad\qquad\qquad \text{(velocity)}. \end{aligned} \tag{27}$$

Here $\tilde{\rho}$ is the total density, ϕ is the pressure, f is the rotation rate, and μ, D are the coefficients of viscosity and heat conduction. The RSSBE have the following nondimensional form:

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + (Ro)^{-1}\vec{\eta} \times \vec{v} + \overline{P}\nabla\phi + \Gamma\rho\vec{e}_3 + \vec{v}\cdot\nabla\vec{v} - (Re)^{-1}\Delta\vec{v} &= 0, \\ \frac{\partial \rho}{\partial t} - (\Gamma)^{-1}(Fr)^{-2}w + \vec{v}\cdot\nabla\rho - (Re)^{-1}(Pr)^{-1}\Delta\rho &= 0, \\ \operatorname{div}\vec{v} &= 0, \end{aligned} \tag{28}$$

with the nondimensional numbers

$$\begin{aligned} Ro &= \frac{U}{Lf} \quad \text{(Rossby)}, & Fr &= \frac{U}{LN} \quad \text{(Froude)}, \\ \overline{P} &= \frac{p}{\rho_b U^2} \quad \text{(Euler)}, & Re &= \frac{\rho_b UL}{\mu} \quad \text{(Reynolds)}, \\ Pr &= \frac{\mu}{\rho_b D} \quad \text{(Prandtl)}, & \Gamma &= \frac{BgL}{U^2}. \end{aligned} \tag{29}$$

In (29) the coefficient $\rho_b B$ measures the strength of the density perturbations and $N = (gb/\rho_b)^{1/2}$ is the constant buoyancy frequency. Here we have picked the same length scale for both the horizontal and vertical dimensions which we utilized in defining both the Rossby and Froude numbers above. The model numerical simulations for mesoscale phenomena of Bartello (1995), Metais *et al.* (1994), Herring and Metais (1989), Riley *et al.* (1981), and Ramsden and Holloway (1992) all pick equal horizontal and vertical length scales. For simplicity in exposition, a similar isotropic nondimensionalization is utilized in (28); anisotropic effects in the vertical can be incorporated readily in the theory developed here (Majda and Embid, 1995).

The pressure ϕ can be eliminated from RSSBE in standard fashion by computing the divergence of the momentum equation in (28) together with the incompressibility constraint and then solving the resulting elliptic equation for the pressure ϕ to get

$$\overline{P}\nabla\phi = \nabla\Delta^{-1}\left((Ro)^{-1}\vec{\eta}\cdot\vec{\omega} - \Gamma\frac{\partial\rho}{\partial x_3} - \operatorname{div}(\vec{v}\cdot\nabla\vec{v})\right), \tag{30}$$

where $\vec{\eta}\cdot\vec{\omega} = \omega_3 = \partial v_2/\partial x - \partial v_1/\partial y$ is the vertical component of vorticity. By inserting (30) in (28), we eliminate the pressure variable and obtain the

Nonlocal Form of Rotating Boussinesq Equations

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + (Ro)^{-1}\vec{\eta}\times\vec{v} + \Gamma\rho\vec{e}_3 + \nabla\Delta^{-1}\left((Ro)^{-1}\vec{\eta}\cdot\vec{\omega} - \Gamma\frac{\partial\rho}{\partial x_3}\right) & \\ + \vec{v}\cdot\nabla\vec{v} - \nabla\Delta^{-1}(\operatorname{div}(\vec{v}\cdot\nabla\vec{v})) - (Re)^{-1}\Delta\vec{v} &= 0, \\ \frac{\partial\rho}{\partial t} - (\Gamma)^{-1}(Fr)^{-2}w + \vec{v}\cdot\nabla\rho - (Re)^{-1}(Pr)^{-1}\Delta\rho &= 0. \end{aligned} \tag{31}$$

We introduce the variable $\vec{u} = (\vec{v}, \rho)$ so that the equations in (31) have the abstract form

$$\begin{gathered} \frac{\partial\vec{u}}{\partial t} + \mathcal{L}(\vec{u}) + \mathcal{B}(\vec{u},\vec{u}) - \mathcal{D}(\vec{u}) = 0, \\ \vec{u}|_{t=0} = \vec{u}_0 \end{gathered} \tag{32}$$

with

$$\begin{aligned} \mathcal{L}(\vec{u}) &= \begin{pmatrix} (Ro)^{-1}\vec{\eta}\times\vec{v} + \Gamma\rho\vec{e}_3 + \nabla\Delta^{-1}((Ro)^{-1}\vec{\eta}\cdot\vec{\omega} - \Gamma(\partial\rho/\partial x_3)) \\ -(\Gamma)^{-1}(Fr)^{-2}w \end{pmatrix}, \\ \mathcal{B}(\vec{u},\vec{u}) &= \begin{pmatrix} \vec{v}\cdot\nabla\vec{v} - \nabla\Delta^{-1}(\operatorname{div}(\vec{v}\cdot\nabla\vec{v})) \\ \vec{v}\cdot\nabla\rho \end{pmatrix}, \\ \mathcal{D}(\vec{u}) &= \begin{pmatrix} (Re)^{-1}\Delta\vec{v} \\ (Re)^{-1}(Pr)^{-1}\Delta\rho \end{pmatrix}. \end{aligned} \tag{33}$$

If the initial velocity field $\vec{v}_0$ is incompressible, then it follows that the solution of (32) automatically satisfies div $\vec{v} \equiv 0$ for all times (Embid and Majda, 1996, 1997), so we view (32) as acting on the space of functions $\vec{u} = (\vec{v}, \rho)$ with div $\vec{v} = 0$.

2.2. The Low Froude Number and Finite Rossby Number and Low Froude Number, Low Rossby Number Limits for the Rotating Stably Stratified Boussinesq Equations

For the two distinguished limits in the title of this subsection, we pick the following balance of the six nondimensional numbers in (29) in straightforward fashion:

$$\begin{array}{ll} \text{Low } Fr, \ \text{finite } Ro & \text{Low } Fr, \text{ low } Ro \\ Fr = \varepsilon \ll 1 & Fr = \varepsilon \ll 1 \\ Ro = O(1) & Ro = \varepsilon/F \ll 1 \\ \overline{P} = \overline{P}_0\,\varepsilon^{-1} & \overline{P} = \overline{P}_0\,\varepsilon^{-1} \\ \Gamma = \varepsilon^{-1} & \Gamma = \varepsilon^{-1} \\ Re \geq O(1) & Re \geq O(1) \\ Pr \geq O(1) & Pr \geq O(1) \end{array} \tag{34}$$

In each case we arrive at the abstract scaled equations

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + \varepsilon^{-1}\mathcal{L}_{\mathrm{F}}(\vec{u}) + \mathcal{L}_{\mathrm{S}}(\vec{u}) + \mathcal{B}(\vec{u}, \vec{u}) - \mathcal{D}(\vec{u}) = 0, \\ \vec{u}|_{t=0} = \vec{u}_0(x), \end{aligned} \tag{35}$$

where the quadratically nonlinear terms and the diffusion terms in (35) have already been given in (33). The slow and fast terms for the linear operators $\mathcal{L}_{\mathrm{F}}(\vec{u})$ and $\mathcal{L}_{\mathrm{S}}(\vec{u})$ are different in the two distinguished limits and are readily computed from (33), (34) and the identity,

$$\mathcal{L}(\vec{u}) = \varepsilon^{-1}\mathcal{L}_{\mathrm{F}}(\vec{u}) + \mathcal{L}_{\mathrm{S}}(\vec{u}).$$

We do not write these formulas explicitly here (see Embid and Majda, 1997) but instead compare the dispersion relations for the skew-symmetric fast wave linear operator

$$\frac{\partial \vec{u}}{\partial t} + \varepsilon^{-1}\mathcal{L}_{\mathrm{F}}(\vec{u}) = 0 \tag{36}$$

acting on the vector-valued functions $\vec{u} = \binom{\vec{v}}{\rho}$ with div $\vec{v} = 0$. Solutions of the form $\vec{u} = \exp(i\vec{k}\cdot\vec{x} - i\omega(\vec{k})t)\vec{r}$ satisfy (36) provided $\omega(\vec{k})$ satisfies the dispersion relation

$$\omega(\omega^2 - \varepsilon^{-2}\Omega^2) = 0 \tag{37}$$

with $\Omega(\vec{k})$ in the two situations given by

$$\begin{array}{ll} \text{Low } Fr, \text{ finite } Ro & \text{Low } Fr, \text{ low } Ro(\vec{\eta} = \vec{e}_3) \\ \Omega = \dfrac{|\vec{k}_{\mathrm{H}}|}{|\vec{k}|} & \Omega = \dfrac{\sqrt{|\vec{k}_{\mathrm{H}}|^2 + F^2 k_3^2}}{|\vec{k}|}. \end{array} \tag{38}$$

This linear analysis in (37) and (38) indicates that the slow waves are different and more plentiful in the low Froude, finite Rossby limit since one root $\omega = 0$ is associated with the vortical modes but in addition there are slow gravity waves with $\vec{k}_{\mathrm{H}} \equiv 0$ where $\Omega \equiv 0$; these slow gravity waves are vertically sheared horizontal motions (see Embid and Majda, 1997). On the other hand, in the low Froude and low Rossby number limit, we see from (38) that Ω is strictly bounded away from zero and all gravity waves are fast waves in this distinguished limit.

With the abstract structure in (35) which we have just developed for the two distinguished limits in (34), it is evident that the same general averaging principle, as developed in (10) and (16) from the Introduction, applies to the two distinguished limits. In both cases, there is nonlinear Rossby adjustment as described in the Introduction for RSWE and the limiting slow dynamics proceeds independently of the fast gravity waves in the limit $\varepsilon \ll 1$. The most elegant way to see this is through the systematic use of Ertel's theorem (see Embid and Majda, 1997). However, as indicated earlier in our discussion above, the limiting slow dynamics is very different for these two distinguished limits.

2.3. The Limiting Dynamics in the Two Regimes

The slow dynamics in the low Froude and low Rossby number distinguished limit is given by the familiar stratified quasi-geostrophic equations (Pedlosky, 1987)

$$\begin{aligned} \vec{v}_{\mathrm{H}} &= \nabla^{\perp}\psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right), \\ q &= \Delta_H \psi + F^2 \frac{\partial^2 \psi}{\partial z^2}, \\ \frac{D^{\mathrm{H}} q}{Dt} &= (Re)^{-1} \Delta \left(\Delta_{\mathrm{H}} \psi + (Pr)^{-1} F^2 \frac{d^2 \psi}{dz^2} \right) \end{aligned} \tag{39}$$

where $D^{\mathrm{H}}/Dt = \partial/\partial t + \vec{v}_{\mathrm{H}}(\vec{x}, t) \cdot \nabla_{\mathrm{H}}$ and $\Delta_{\mathrm{H}}\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2$.

On the other hand, the slow dynamics in the low Froude and fixed Rossby number distinguished limit is given through the velocity decomposition

$$\begin{aligned} \vec{v}_{\mathrm{H}} &= \vec{V}_{\mathrm{H}} + \nabla^{\perp}\psi, \\ \vec{V}_{\mathrm{H}} &= \vec{V}_{\mathrm{H}}(z, t), \\ \psi &= \psi(\vec{x}_{\mathrm{H}}, z, t), \end{aligned} \tag{40}$$

and the dynamical equations

$$\begin{aligned} \frac{\partial}{\partial t}\vec{V}_{\mathrm{H}} + (Ro)^{-1}\vec{V}_{\mathrm{H}}^{\perp} &= (Re)^{-1}\frac{\partial^2}{\partial z^2}\vec{V}_{\mathrm{H}}, \\ \frac{\partial \omega}{\partial t} + \vec{V}_{\mathrm{H}} \cdot \nabla_{\mathrm{H}}\omega + J_{\mathrm{H}}(\psi, \omega) &= (Re)^{-1}\Delta_{\mathrm{H}}\omega + (Re)^{-1}\frac{\partial^2 \omega}{\partial z^2}, \\ \Delta_{\mathrm{H}}\psi &= \omega \end{aligned} \tag{41}$$

where $J_{\mathrm{H}}(\psi, \omega) = \nabla^{\perp}\psi \cdot \nabla_{\mathrm{H}}\omega$ as well as the decoupled equations

$$\begin{aligned} w &\equiv 0, \\ \frac{\partial \rho}{\partial t} &= (Re)^{-1}(Pr)^{-1}\frac{\partial^2 \rho}{\partial z^2}. \end{aligned} \tag{42}$$

Of course, ω in (41) is the vertical component of vorticity. The limiting slow dynamics in (41) contain the vertically sheared horizontal motions, $\vec{V}_{\mathrm{H}}(z, t)$, which represent the low-speed gravity waves with wave numbers satisfying $|\vec{k}_{\mathrm{H}}| = 0$ and discussed earlier in (37) and (38).

Majda and Grote (1997) have developed exact solutions of the equations in (41) involving a periodic array of dipole vortices in a weak shear which qualitatively captures the basic features of the remarkable vertical collapse in decaying strongly stratified flows which has been observed in recent laboratory experiments by Fincham *et al.* (1996) (see Maxworthy, 1997). Finally, we remark that the general slow dynamics in (41) cannot be recovered by setting $Ro = Fr/F$ in the quasi-geostrophic equations in (39) and letting $F \to 0$ in a formal fashion; in fact, the limit $F \to 0$ in (39) yields the special case of (41) with $\vec{V}_{\mathrm{H}}(z, t) \equiv 0$ so that the vertical shears are completely missed by this limiting procedure.

There are other important differences in the reduced dynamics in these two cases involving the slow–fast–fast and fast–fast–fast resonances. Also, for interpreting laboratory experiments in stratified saltwater solutions with Prandtl number $Pr = O(200)$, there are interesting differences in the two regimes for behavior of the reduced dynamics at large Prandtl numbers. The interested reader can consult Embid and Majda (1997) for a detailed discussion of all of these issues. The quasi-geostrophic equations in (39) have also been derived rigorously by Bourgeois and Beale (1994) as a singular limit from the RSSBE provided the initial data is in geostrophic and hydrostatic balance through a classical approach generalizing the procedure of Klainerman and Majda (1981, 1982) and Majda (1984).

3. Asymptotic Solutions of the Rapidly Rotating Shallow-Water Equations over Many Large-Scale Eddy Turnover Times

We consider the possibility of constructing asymptotic solutions of the rapidly RSWE from (9) over many large-scale eddy turnover times for $\varepsilon \ll 1$. Thus, we introduce the long time scale $T = \varepsilon t$ and discuss formally valid long-time asymptotic solutions of rapidly RSWE which involve, for $\varepsilon \ll 1$, the three time scales: (1) $\tau = \varepsilon^{-1}t$, the gravity wave scale, (2) $\tau = O(1)$, the large-scale turnover time, and (3) $T = \varepsilon t$, many large-scale eddy turnover times. Why do we do this? There are two fundamental theoretical issues which we would like to address with such an approach:

1. While, as discussed earlier, there is no effect of the gravity modes on the quasi-geostrophic modes for $O(1)$ large-scale advective times for $\varepsilon \ll 1$, is there a feedback of the gravity modes on the quasi-geostrophic modes over many $O(\varepsilon t)$ large-scale turnover times? (43)
2. Numerical simulations with dissipation and decaying turbulence such as those of Bartello (1995) indicate that with strong rotation and stratification, over many eddy turnover times, the ageostrophic modes decay much more rapidly than the potential vortical modes. Is there a robust elementary nonlinear interaction between vortical modes and gravity modes for the RSWE which gives insight into the above process? (44)

Here we briefly report on work in Callet (1997) and Callet and Majda (1997) which provides some insight into both of the issues described in (43) and (44). In this work, simple initial data consisting of resonant triads and resonant quartets of waves are considered where one or two of the modes are quasi-geostrophic while the remaining modes are inertio-gravity waves. All of the various cases of such interactions are considered in Callet (1997) and Callet and Majda (1997).

As regards the basic issue from (43), in all of these cases and even over many eddy turnover times, i.e., for times $T = \varepsilon t$ with $\varepsilon \ll 1$, the quasi-geostrophic modes proceed independently from the inertio-gravity modes without any feedback; the inertio-gravity modes are affected by the vortical modes and exchange energy through various interactions. Thus, for these special resonant configurations, nonlinear Rossby adjustment as described for rapidly RSWE in the Introduction remains valid over many large-scale turnover times. This behavior for RSWE contrasts strongly with the resonant behavior of surface gravity waves interacting with vortical waves in deep water as developed recently by Milewski and Benney (1995); in the situation considered by Milewski and Benney, for simple resonant quartets, the gravity modes have feedback on the vortical modes on long time scales through a generalized Stokes drift. Next, we discuss the issue in (44) in somewhat more detail.

3.1. Enhanced Gravity Wave Dissipation through Catalytic Interaction with the Potential Vortical Modes

There is a robust family of resonant quartets (Callet and Majda, 1997) for rapidly RSWE which yield enhanced gravity wave dissipation at moderate wave numbers through catalytic interaction via the potential vortical modes. Similar families of resonant quartets also occur in RSSBE (Callet and Majda, 1997). To construct these robust resonant quartets, choose three gravity waves at wave numbers $\vec{k}_1, \vec{k}_2, \vec{k}_3$ so that

$$\omega(\vec{k}_1) + \omega(\vec{k}_2) + \omega(\vec{k}_3) = 0, \tag{45}$$

where

$$\omega(\vec{k}_j) = \alpha_j(1 + F^{-1}|\vec{k}_j|^2)^{1/2}, \qquad \alpha_j = \pm 1.$$

The conditions in (45) imply that one of the waves will be on the opposite branch from the other two, i.e., if $\alpha_1 = -\alpha_2 = -\alpha_3$, then the conditions in (45) become

$$|\omega(\vec{k}_1)| = |\omega(\vec{k}_2)| + |\omega(\vec{k}_3)|.$$

There are plentiful families of inertio-gravity waves satisfying (45). We claim that such families trivially interact in a resonant quartet with a potential vortical mode; to achieve this, we simply define the wavelength of the potential vortical mode, $\vec{k}_0$, by

$$\vec{k}_0 + \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0. \tag{46}$$

Since for all the vortical modes, $\omega^0(\vec{k}_0) = 0$, the conditions in (45) and (46) guarantee that these waves form a resonant quartet of three gravity waves and one vortical mode.

The asymptotic equations (see Callet and Majda, 1997) for the amplitudes of this resonant quartet over many turnover times for the rapidly RSWE with, for example, fourth-order hyperviscosity dissipation are given by

$$\begin{aligned}
\frac{d\sigma_0}{dT} &= -\left(\frac{|\vec{k}_0|}{|\vec{k}_\mathrm{d}|}\right)^4 \sigma_0,\\
\frac{d\sigma_1}{dT} + i\sum_{j=0}^{3}\Gamma_{jj1}|\sigma_j|^2\sigma_1 + \omega_1\Gamma\sigma_0^*\sigma_2^*\sigma_3^* &= -\left(\frac{|\vec{k}_1|}{|\vec{k}_\mathrm{d}|}\right)^4 \sigma_1,\\
\frac{d\sigma_2}{dT} + i\sum_{j=0}^{3}\Gamma_{jj2}|\sigma_j|^2\sigma_2 + \omega_2\Gamma\sigma_0^*\sigma_1^*\sigma_3^* &= -\left(\frac{|\vec{k}_2|}{|\vec{k}_\mathrm{d}|}\right)^4 \sigma_2,\\
\frac{d\sigma_3}{dT} + i\sum_{j=0}^{3}\Gamma_{jj3}|\sigma_j|^2\sigma_3 + \omega_3\Gamma\sigma_0^*\sigma_1^*\sigma_2^* &= -\left(\frac{|\vec{k}_3|}{|\vec{k}_\mathrm{d}|}\right)^4 \sigma_3.
\end{aligned} \tag{47}$$

In (47) σ_0 is the complex amplitude of the vortical mode while σ_j, $j = 1, 2, 3$, are the complex amplitudes of the three gravity modes satisfying (45) with $\omega_j = \omega(\vec{k}_j)$. The time scale $T = \varepsilon t$ involves many large-scale eddy turnover times while the real numbers Γ_{jji} and Γ are interaction coefficients with complicated algebraic expressions (Callet, 1997) with a detailed form which is not needed for the discussion here. The quantity $|\vec{k}_d|^{-4}$ measures the coefficient of dissipation of the hyperviscosity operator.

From (47) we see that for these resonant quartets the potential vortical mode proceeds independently from the gravity modes but when σ_0 is nonzero, the potential vortical mode acts as a catalyst for energy exchange among the three gravity modes through quadratic interaction among these gravity waves. In other words, once σ_0 is nonzero, the energy exchange among the gravity modes is through what are effectively three-wave resonant interactions but on a longer time scale; this fact is interesting because direct three-wave resonances among gravity waves are impossible for RSWE (Embid and Majda, 1996).

We claim that the resonant quartet mechanism in (45)–(47) is very effective in promoting enhanced dissipation of moderate wavelength inertio-gravity waves through catalytic interaction by a moderate wavelength vortical mode with a shorter wavelength gravity mode. This is surprising at first glance because, according to linear theory, the shorter wavelength gravity mode dissipates at a much faster rate due to hyperviscosity. A general study of the robustness of this mechanism including scaling behavior and possible statistical implications is presented in Callet and Majda (1997). Here we simply present an explicit example demonstrating this effect.

We choose $F \equiv 1$, and $|\vec{k}_\mathrm{d}| = 8$ in (45) and (46). We pick three gravity waves with frequencies 4, 5, and -9 so that (45) is automatically satisfied. The corresponding wave numbers, wavelengths, and frequencies

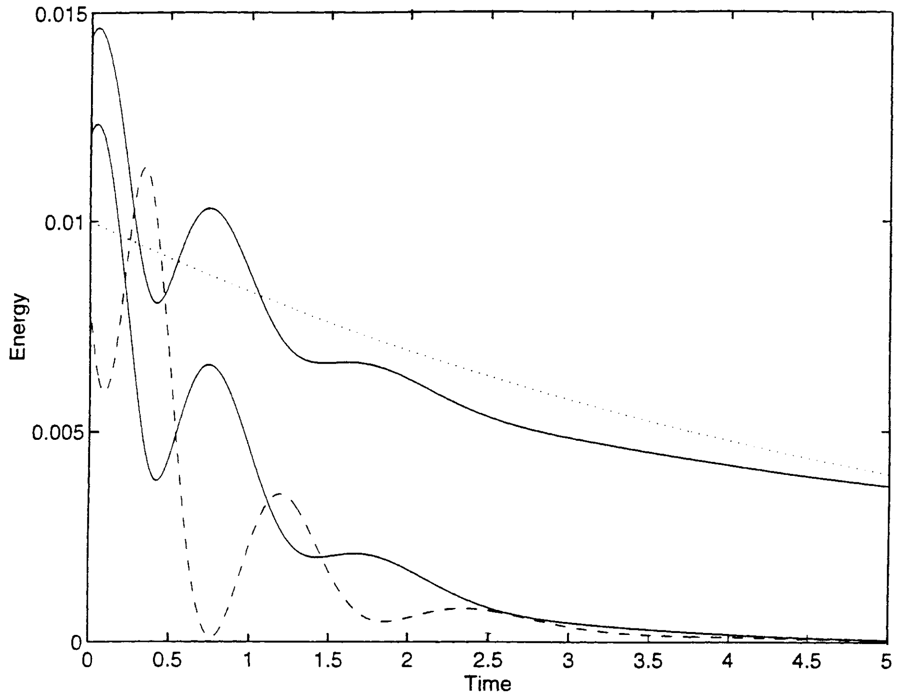

Figure 1. The energy of the moderate wavelength gravity modes $\vec{k}_1, \vec{k}_2$ (solid lines), the short wavelength gravity mode $\vec{k}_3$ (dashed line), and the potential vortical mode $\vec{k}_0$ (dotted line) as functions of time for the numerical solution of (47).

for this resonant quartet satisfying (45) and (46) are

$$\begin{array}{lll} \vec{k}_0 = (-0.93, -4.31), & |\vec{k}_0| = 4.41, & \omega_0 = 0, \\ \vec{k}_1 = (3.87, 0), & |\vec{k}_1| = 3.87, & \omega_1 = 4, \\ \vec{k}_2 = (4.90, 0), & |\vec{k}_2| = 4.90, & \omega_2 = 5, \\ \vec{k}_3 = (-7.84, 4.31), & |\vec{k}_3| = 8.94, & \omega_3 = -9. \end{array} \tag{48}$$

We note that the vortical mode with $\vec{k}_0$ and the two gravity modes with $\vec{k}_1, \vec{k}_2$ have moderate wavelength while the gravity mode $\vec{k}_3$ has amplitude within the region of significant dissipation by hyperviscosity since $|\vec{k}_\mathrm{d}| = 8$. We pick an initial amplitude of 0.10 for the vortical mode, with the slightly larger amplitudes 0.12 and 0.11 for the gravity modes with wave numbers $\vec{k}_1$ and $\vec{k}_2$, respectively, while we use the smaller amplitude 0.09, for the gravity mode with $\vec{k}_3$.

In Figure 1 we present the time history of the numerical solution of (47) with this initial data. The two solid lines depict the energy in the wavelength $\vec{k}_1$ and $\vec{k}_2$ gravity modes, the dashed line gives the energy in the $\vec{k}_3$ gravity mode, and the dotted line gives the energy in the vortical mode. The numerical solution clearly demonstrates remarkable enhanced dissipation of the moderate wavelength $\vec{k}_2$ gravity mode through catalytic nonlinear interaction, at a rate comparable with that of the higher wavelength $\vec{k}_3$ gravity mode. Of course, beyond times $t \gtrsim 2$, the amplitudes of both σ_2 and σ_3 have diminished so significantly that the nonlinear interaction in (47) is effectively switched off. Nevertheless, the potential vortical mode has the largest amplitude of the four waves for all times $t \gtrsim 1$ which are depicted in Figure 1. This is the basic mechanism developed in Callet and Majda (1997).

4. Some Open Problems in Fast Wave Averaging for Geophysical Flow

Besides the issues regarding fast wave averaging for geophysical flows in the previous sections, there are at least three other important topics where a more sophisticated mathematical analysis could significantly improve and clarify the understanding of these problems:

1. **Reduced Dynamics for Equatorial Waves**. It is well known (Gill, 1982) that frequency scale separation between potential vortical and gravity waves is not strictly valid near the equator; the linearized equatorial shallow water equations include Yanai or mixed-Rossby gravity waves. Is it possible to describe a system of equations with a rigorous derivation which captures the appropriate slow dynamics for solutions of the equatorial shallow-water equations or even the primitive equations? This problem involves the notion of balanced dynamics. The recent work by Warn *et al.* (1995) is an excellent research/expository paper discussing midlatitude balanced dynamics and might provide a useful background reference.

2. **Boundary Conditions and Quasi-Geostrophic Surface Waves**. The classical quasi-geostrophic equations for a rapidly rotating strongly stratified fluid in a half-space or above a sphere involve the quasi-geostrophic equations in the interior (see (39) above) coupled with surface quasi-geostrophic equations for the boundary evolution of thermal fronts (Pedlosky, 1987). Even for initial data in approximate geostrophic and hydrostatic balance, is it possible to give a mathematically precise derivation of the coupled system involving quasi-geostrophic flow and surface quasi-geostrophic thermal fronts? What is the effect of unbalanced initial data for this problem? What about the effects of anisotropy and surface frontogenesis (Pedlosky, 1987)? The papers of Bourgeois and Beale (1994) and Embid and Majda (1997) discussed earlier in Section 2 provide the basic mathematical background but new ideas are needed.

3. **Anelastic Equations for Deep Atmospheric Convection**. There are several different forms for the anelastic equations utilized for deep atmospheric convection and there is a genuine controversy over which version of these equations should be preferred (see several sections of the *Proceedings of the Tenth Conference on Waves and Stability*, 1995). Is it possible to provide a rigorous mathematical justification of the appropriate correct version for these anelastic equations including the effects of moisture? The basic reference, Majda (1984), as well as the work of Schochet (1987), provides a useful mathematical starting point.

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