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ORIGINAL ARTICLE

Rupert Klein · Andrew J. Majda

Systematic multiscale models for deep convection on mesoscales

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Abstract This paper builds on recent developments of a unified asymptotic approach to meteorological modeling [ZAMM, 80:765–777, 2000, SIAM Proc. App. Math. 116, 227–289, 2004], which was used successfully in the development of *Systematic multiscale models for the tropics* in Majda and Klein [J. Atmosph. Sci. 60: 393–408, 2003] and Majda and Biello [PNAS, 101: 4736–4741, 2004]. Biello and Majda [J. Atmosph. Sci. 62: 1694–1720, 2005]. Here we account for typical bulk microphysics parameterizations of moist processes within this framework. The key steps are careful nondimensionalization of the bulk microphysics equations and the choice of appropriate distinguished limits for the various nondimensional small parameters that appear. We are then in a position to study scale interactions in the atmosphere involving moist physics. We demonstrate this by developing two systematic multiscale models that are motivated by our interest in mesoscale organized convection. The emphasis here is on multiple length scales but common time scales. The first of these models describes the short-time evolution of slender, deep convective *hot towers* with horizontal scale ~ 1 km interacting with the linearized momentum balance on length and time scales of (10 km/3 min). We expect this model to describe how convective inhibition may be overcome near the surface, how the onset of deep convection triggers convective-scale gravity waves, and that it will also yield new insight into how such local convective events may conspire to create larger-scale strong storms. The second model addresses the next larger range of length and time scales (10 km, 100 km, and 20 min) and exhibits mathematical features that are strongly reminiscent of mesoscale organized convection. In both cases, the asymptotic analysis reveals how the stiffness of condensation/evaporation processes induces highly nonlinear dynamics. Besides providing new theoretical insights, the derived models may also serve as a theoretical devices for analyzing and interpreting the results of complex moist process model simulations, and they may stimulate the development of new, theoretically grounded sub-grid-scale parameterizations.

Keywords Moist processes · Multiple-scale asymptotics

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R. Klein (✉)
FB Mathematik and Informatik,
Freie Universität Berlin and Potsdam Institute for Climate Impact Research,
Berlin, Germany
E-mail: rupert.klein@zib.de

A. J. Majda
Courant Institute of Mathematical Sciences, New York University, New York, NY, USA
E-mail: jonjon@cims.nyu.edu

1 Introduction

The accurate model-based prediction of moist atmospheric flows remains one of the most demanding challenges in theoretical meteorology. A clear indication is the fact that over recent decades the score of successful predictions for fluid dynamical variables, such as pressure, temperature, and mesoscale wind fields, has remained systematically higher than the prediction score for precipitation. Since water vapor is also by far the most active greenhouse gas, and since clouds affect the radiation balance in addition through albedo effects, the understanding of moist atmospheric flows is also crucial for longer-term predictions from seasons to climate time scales.

There is, in particular, a large number of flow phenomena in the near-tropical atmosphere that involve multiscale interactions of (small-scale) moist processes with larger-scale mean flows. Here is a citation from [11] that supports this point: "The essence of tropical dynamics is the intricate balance between large-scale processes – such as radiative transfer, large-scale waves, monsoons, Hadley and Walker circulations – and the convective dynamics."

Given the success of the authors' recent development of systematic multiscale models for the tropics, [17], in providing promising new hypotheses regarding the origins of the Madden–Julian oscillation, [2,18], an extension of the multiple-scale asymptotic approach from [13–15] to include the effects of moist physics explicitly appears promising. We proceed in this direction in the present paper. In doing so, our goal is to assess the feasibility of such an endeavor rather than to build a fully comprehensive reduced model with respect to the complexities of multiphase moist flows. Thus we restrict ourselves to a relatively simple description of the physics of moist flows called a bulk microphysics closure scheme. Such schemes are employed in theoretical meteorology to describe the development of clouds and precipitation on and above length and time scales of about one hundred meters and a few minutes. The simple closure scheme considered here involves merely three transport equations for the mass fractions of water vapor, cloud water (microscopic aerosol droplets), and precipitation (large droplets). We leave descriptions of nontrivial droplet-size distributions, the ice phase, etc. for future work.

Section 2 provides a detailed summary of the governing equations to be analyzed in this paper. These are the three-dimensional compressible flow equations with gravity and rotation, and they include a version of bulk microphysics parameterizations as proposed, e.g., in [8,12]. Such a model provides a suitable basis for studying processes on length and time scales of $(1\ \text{km},\ 3\ \text{min})$ and above.

Section 3 presents moist atmospheric processes from an asymptotic perspective. Here we nondimensionalize the governing equations and identify various nondimensional parameters – particularly those which characterize the moist processes. These include large Damköhler numbers, indicating rapid transitions between the various moisture species, and a large activation energy parameter in Bolton's Arrhenius-type approximation to the Clausius–Clapeyron relation, (see [6]). Being faced with a system of equations involving a considerable number of small or large parameters, we follow [14] and introduce a series of distinguished limits tying each of these variables to the single small parameter, $\varepsilon \ll 1$, from [14,17], which then serves as the starting point for subsequent asymptotic expansions.

We move on to describe two distinct multiscale regimes for deep convective moist flows that are likely to play central roles in the dynamics of mesoscale convective systems: the smaller-scale regime addresses the short time dynamics of narrow deep convective columns, the building blocks of intermediate-scale convective storms. The second regime describes the interaction of such strong convective events with mesoscale gravity waves and mean flows. The choice of these two regimes is strongly motivated by studies of organized convection in [19–21] on one hand, and of column model parameterizations of deep convection in, e.g., [7], on the other hand. Thus, we consider organized mesoscale convection as a three-scale process, and the two simplified asymptotic multiscale models presented in this paper address pairwise interactions between the three participating spacial scales.

Section 4 provides our first multiscale model for moist processes. We consider the short-time evolution over several minutes of narrow, deep convective columns with characteristic horizontal dimension of ~ 1 km, embedded within a convective-scale (~ 10 km) environment. A summary of the key results of the derivations is given in Sect. 4.1. (See also [3,4] for an earlier analysis of diabatically driven columnar flow of dry air.) We expect this model to describe how convective inhibition may be overcome near the surface, how the onset of deep convection triggers convective-scale gravity waves, and that it will also yield new insight into how such local convective events may conspire to create larger-scale strong storms.

Section 5 considers the next larger range of length and time scales involving convective scales of $\sim 10\ \text{km}/20\ \text{min}$ and their coupling to linearized mesoscale motions with characteristic horizontal scales of

$\sim$ 100 km. Again, we provide a summary of the key results in the separate Sect. 5.1. This model addresses the scale ranges relevant to phenomena of organized mesoscale convection as discussed, e.g., in [19–21].

These simplified asymptotic models may serve as guidelines in the analysis and interpretation of complex moist atmospheric flow simulations. They may also be used in defining specific tests for complex flow models in such a way that the models' capabilities to represent subtle scale interactions can be assessed. Finally, our simplified asymptotic multiscale models may be useful as a new basis for the development of sub-grid-scale parameterizations, such as the column models for deep convection considered, e.g., in [7].

Section 6 draws some conclusions.

1.1 Notation

Preliminary remarks:

1. Plain symbols denote nondimensional quantities.
2. $\varepsilon \ll 1$ is a generic asymptotic expansion parameter, see Sect. 3.1.
3. $(t, \boldsymbol{x}, z)$ with $\boldsymbol{x} = (x, y)$ are the time, and the horizontal and vertical space coordinates, nondimensionalized by $t_{\text{ref}} = h_{\text{scale}}/u_{\text{ref}}$ and $\ell_{\text{ref}} = h_{\text{scale}}$, where $h_{\text{scale}} \approx 10$ km is the atmospheric pressure scale height, and $u_{\text{ref}} \approx 10$ m/s is a typical air flow velocity.

In addition, we adhere to the following notational conventions:

U'	dimensional version of the physical quantity U
$U^{(i)}$	ith scaled term in an asymptotic expansion $U = \sum_{i=0}^{n} \varepsilon^i \, U^{(i)} + o(\varepsilon^n)$
$C^*, q^*, \ldots$	dimensionless constants, generally functions of ϵ
$C^{**}, q^{**}, \ldots$	dimensionless constants, scaled so that $C^{**}, q^{**} = O(1)$ as $\varepsilon \to 0$
$H_{\geq}, H_{>}, \ldots$	Heaviside-type step functions
$\tilde{S}_\theta$	diabatic potential temperature source *unrelated to* latent-heat release
S_θ^q	diabatic potential temperature source *due to* latent-heat release
$v_\perp, \boldsymbol{v}_\parallel$	projections: $v_\perp = \boldsymbol{v} \cdot \boldsymbol{k}$, $\boldsymbol{v}_\parallel = (\boldsymbol{1} - \boldsymbol{k} \circ \boldsymbol{k})\, \boldsymbol{v}$ with $\boldsymbol{k}$ the vertical unit vector

2 Governing equations

In this paper, we analyze the equations for compressible, moist atmospheric flows on scales of 1 km and above in two asymptotic scaling regimes. Here we summarize the balance equations for mass, momentum, and energy (potential temperature or entropy), as well as bulk microphysics representations of moist processes that are valid for spacio-temporal scales larger or equal to 1 km and 150 s as used in this paper. These representations are modeled after the scheme in [8] with some modifications that we will explain as we go along.

2.1 Governing equations

We work with nondimensional expressions as far as possible. The nondimensionalization uses standard reference values for pressure, density, and velocity, i.e.,

$$p_{\text{ref}} = 10^5 \text{ Pa}\,, \quad \rho_{\text{ref}} = 1.25 \text{ kg/m}^3\,, \quad u_{\text{ref}} = 10 \text{ m/s}\,, \tag{1}$$

and the deep convective scales for scaling the space and time coordinates,

$$\ell_{\text{ref}} = h_{\text{scale}} = \frac{p_{\text{ref}}}{g\, \rho_{\text{ref}}} \approx 10 \text{ km}\,, \quad t_{\text{ref}} = \frac{h_{\text{scale}}}{u_{\text{ref}}} \approx 20 \text{ min} \sim 1000 \text{ s}\,. \tag{2}$$

Here g denotes the acceleration of gravity, and h_{scale} is the pressure scale height. Except for these reference quantities we will mark dimensional variables by a prime superscript from here on.

Mass, momentum, energy balances

$$
\begin{aligned}
\rho_t + \nabla_\| \cdot (\rho \boldsymbol{u}) + (\rho w)_z &= 0 \\
\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla_\| \boldsymbol{u} + w \boldsymbol{u}_z + \frac{1}{\mathrm{Ro_B}} (\boldsymbol{\Omega} \times \boldsymbol{v})_\| + \frac{1}{\mathrm{M}^2} \frac{1}{\rho} \nabla_\| p &= \boldsymbol{D_u} \\
w_t + \boldsymbol{u} \cdot \nabla_\| w + w w_z + \frac{1}{\mathrm{Ro_B}} (\boldsymbol{\Omega} \times \boldsymbol{v})_\perp + \frac{1}{\mathrm{M}^2} \frac{1}{\rho} p_z &= D_w - \frac{1}{\overline{\mathrm{Fr}}^2} \\
\theta_t + \boldsymbol{u} \cdot \nabla_\| \theta + w \theta_z &= D_\theta + S_\theta \, .
\end{aligned}
\tag{3}
$$

Here $\rho, \boldsymbol{u}, w, \theta$ are the density, horizontal and vertical flow velocities, and the potential temperature, respectively, and the pressure p is related to ρ, θ through the thermodynamic equation of state

$$
p = (\rho\theta)^\gamma \tag{4}
$$

where γ is the (constant) isentropic exponent. $\boldsymbol{\Omega}$ is the vector of Earth rotation, and the subscripts $_\perp, _\|$ indicate projections onto the vertical direction and the horizontal tangent plane, respectively.

As nondimensional characteristic numbers appear the Mach (M), barotropic Froude ($\overline{\mathrm{Fr}}$), and the bulk microscale Rossby ($\mathrm{Ro_B}$) numbers,

$$
\mathrm{M} = \frac{u_{\mathrm{ref}}}{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}} = \overline{\mathrm{Fr}} = \frac{u_{\mathrm{ref}}}{\sqrt{g\, h_{\mathrm{scale}}}} \sim \frac{1}{30}\,, \quad \text{and} \quad \mathrm{Ro_B} \sim 10\,, \tag{5}
$$

respectively.

The terms $\boldsymbol{D_u}, D_w, D_\theta$ represent the effects of turbulent and molecular transport, for which we will assume standard gradient transport closures where needed. On the length and time scales considered, here we may safely assume that these terms are not dominant or singular (see the turbulent boundary layer analysis in [15]). The potential temperature source term

$$
S_\theta = \frac{\gamma - 1}{\gamma} \frac{\theta}{p} S_{\rho e} \tag{6}
$$

is directly proportional to the source of internal and kinetic energy, $S_{\rho e}$, which itself is a superposition of radiative sources, sources from latent-heat conversion, etc. As we concentrate on the effects of latent-heat conversion in this paper, we will express the related contributions to S_θ in terms of the bulk microphysics models for moist processes in (9) below.

To describe these moist processes, we adopt a somewhat modified version of the bulk microphysics parameterization from [8]. Let the mixing ratio of some species y be defined as the density ratio ρ_y/ρ_d, i.e., as the ratio of the species' density versus that of dry air. Then we introduce *scaled* mixing ratios $q_{\mathrm{v}}, q_{\mathrm{c}}, q_{\mathrm{r}}$ of water vapor, cloud water aerosol, and precipitating water, respectively, via

$$
q_y = \frac{1}{q_{\mathrm{vs}}^*} \frac{\rho_y}{\rho_d} \qquad \text{with} \qquad y \in \{v, c, r\} \quad \text{and} \quad q_{\mathrm{vs}}^* = \left(\frac{\rho_{v,\mathrm{sat}}}{\rho_d} \right)_{\mathrm{ref}} . \tag{7}
$$

Here q_{vs}^* is the saturation value for the unscaled water vapor mixing ratio at reference conditions, $p_{\mathrm{ref}}, \rho_{\mathrm{ref}}$. The reason for introducing this scaling is that: (i) $q_{\mathrm{vs}}^* \sim 2 \times 10^{-2}$ is quite small, and (ii) it sets the order of magnitude for all the water variables, because water vapor provides the main water reservoir. As a consequence of the scaling, we may expect $q_{\mathrm{v}}, q_{\mathrm{r}}, q_{\mathrm{c}}$ *as defined in* (7) to be roughly of order unity in nearly saturated, cloud-forming, precipitating air. These definitions lead to the following relationships.

Moisture balances

$$
\begin{aligned}
q_{\mathrm{v},t} + \boldsymbol{u} \cdot \nabla_\| q_{\mathrm{v}} + w q_{\mathrm{v},z} \qquad\qquad &= (C_{\mathrm{ev}} - C_{\mathrm{d}}) \qquad\quad + D_{q\mathrm{v}} \\
q_{\mathrm{c},t} + \boldsymbol{u} \cdot \nabla_\| q_{\mathrm{c}} + w q_{\mathrm{c},z} \qquad\qquad &= (C_{\mathrm{d}} - C_{\mathrm{ac}} - C_{\mathrm{cr}}) \ + D_{q\mathrm{c}} \\
q_{\mathrm{r},t} + \boldsymbol{u} \cdot \nabla_\| q_{\mathrm{r}} + w q_{\mathrm{r},z} + \tfrac{1}{\rho} (\rho q_{\mathrm{r}} V_{\mathrm{T}})_z &= (C_{\mathrm{ac}} + C_{\mathrm{cr}} - C_{\mathrm{ev}}) + D_{q\mathrm{r}} \, .
\end{aligned}
\tag{8}
$$

Here again $D_{q\mathrm{v}}, D_{q\mathrm{c}}, D_{q\mathrm{r}}$ describe (turbulent) transport, whereas $C_{\mathrm{ev}}, C_{\mathrm{d}}, C_{\mathrm{ac}}, C_{\mathrm{cr}}$ are the rates of evaporation of rain water, the condensation of water vapor to cloud water (and the inverse evaporation process), the autoconversion of cloud water into rainwater by accumulation of microscopic droplets, and the collection of cloud water by the falling rain, respectively.

2.2 Explicit representations of the source terms

As in (7), a $(\cdot)^*$ superscript will indicate scalar constants as they appear directly from nondimensionalization in the sequel. In the later asymptotic analyses, these quantities may still be functions of our generic expansion parameter ε. We will make this more explicit where needed by employing a double star, $(\cdot)^{**}$, to denote rescaled constants that are $O(1)$ as $\varepsilon \to 0$.

The conversion processes of gaseous water vapor to liquid water and vice versa are associated with the release and/or absorption of latent heat. Accordingly, we separate the moisture-related contribution from other diabatic effects by decomposing the source term, S_θ, in the potential temperature evolution equation as follows,

$$S_\theta = \tilde{S}_\theta + S_\theta^q \quad \text{where } S_\theta^q = \frac{\gamma - 1}{\gamma} \frac{\theta}{p} L^* q_{\mathrm{vs}}^* \left(C_{\mathrm{d}} - C_{\mathrm{ev}}\right). \tag{9}$$

Here $L^* = L/(p_{\mathrm{ref}}/\rho_{\mathrm{ref}})$, and L is the latent heat per unit mass of water vapor, which we assume to be constant for simplicity from here on.

In defining condensation/evaporation of cloud water, it is often assumed in cloud microphysics parameterizations that the vapor-to-cloud water conversion is instantaneous, i.e., that *either* the air is saturated, such that the water vapor content matches its saturation value, $q_{\mathrm{v}} = q_{\mathrm{vs}}(\theta, p)$, and cloud water droplets can exist with $q_{\mathrm{c}} \geq 0$, *or* the air is undersaturated, i.e., $q_{\mathrm{v}} < q_{\mathrm{vs}}$, in which case $q_{\mathrm{c}} \equiv 0$. Rather than *assuming* this limiting behavior from the outset, we will demonstrate here how it may be *derived* in a consistent asymptotic framework given large but finite condensation rates. This is the main deviation of the present bulk microphysics description from the scheme in [8].

Bulk microphysics closure

$$\begin{aligned}
C_{\mathrm{d}} &= C_{\mathrm{d}}^* \left(q_{\mathrm{v}} - q_{\mathrm{vs}}\right) H_\circ(q_{\mathrm{c}}, q_{\mathrm{v}}, q_{\mathrm{vs}})(q_{\mathrm{c}} + q_{cn}^*)\\
C_{\mathrm{ev}} &= -C_{\mathrm{ev}}^* \frac{p}{\rho} \left(q_{\mathrm{v}} - q_{\mathrm{vs}}\right) H_>(q_{\mathrm{r}}) q_{\mathrm{r}}^{1/2+\delta^*}\\
C_{\mathrm{cr}} &= C_{\mathrm{cr}}^* \, q_{\mathrm{c}} q_{\mathrm{r}}^{(1+\alpha^*)}\\
C_{\mathrm{ac}} &= C_{\mathrm{ac}}^* \max\left(0, q_{\mathrm{c}} - q_{\mathrm{c}}^*\right)
\end{aligned} \tag{10}$$

Here $C_{\mathrm{d}}^*, C_{\mathrm{ev}}^*, C_{\mathrm{cr}}^*, C_{\mathrm{ac}}^*$ are dimensionless rate constants, i.e., Damköhler numbers, δ^*, α^* are exponents close to unity, q_{cn}^* represents the likelihood of onset of condensation as determined by the available amount of condensation nuclei, and q_{c}^* is a threshold value for the cloud-water mixing ratio beyond which autoconversion of cloud water into precipitation becomes active. Using the biased Heaviside step functions

$$H_\geq(q) = \begin{cases} 1 & (q \geq 0) \\ 0 & \text{otherwise}, \end{cases} \qquad H_>(q) = \begin{cases} 1 & (q > 0) \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

we define the switching function $H_\circ(\cdot, \cdot, \cdot)$ via

$$H_\circ(q_{\mathrm{c}}, q_{\mathrm{v}}, q_{\mathrm{vs}}) = H_\geq(q_{\mathrm{v}} - q_{\mathrm{vs}}) H_\geq(q_{\mathrm{c}}) + H_>(q_{\mathrm{vs}} - q_{\mathrm{v}}) H_>(q_{\mathrm{c}}). \tag{12}$$

With this switch we achieve the desired behavior in cloudless air ($q_{\mathrm{c}} = 0$): There will be positive condensation rates in oversaturated air, ($q_{\mathrm{v}} > q_{\mathrm{vs}}$), but evaporation ceases in undersaturated air ($q_{\mathrm{v}} < q_{\mathrm{vs}}$) in this case.

The saturation vapor mixing ratio, q_{vs}, is given by [6],

$$q_{\mathrm{vs}}(\theta, p) = \frac{1}{q_{\mathrm{vs}}^*} \frac{E e_s(\theta, p)}{p - e_s(\theta, p)} \tag{13}$$

with the ratio of the dry air and water vapor gas constants being

$$E = \frac{R_d}{R_v}, \tag{14}$$

and the saturation vapor pressure according to Bolton's formula

$$e_s(\theta, p) = e_s^* \exp\left(A^* \frac{T(\theta, p) - T_0^*}{T_1^* + (T(\theta, p) - T_0^*)}\right). \tag{15}$$

Here, the temperature, T, obeys

$$T(\theta, p) = \theta\, p^{\frac{\gamma-1}{\gamma}}\,. \tag{16}$$

Notice that, by our definition in (7), q_{vs}^* is determined via $q_{\mathrm{vs}}(1, 1) = 1$, i.e.,

$$q_{\mathrm{vs}}^* = \frac{E e_s^*}{1 - e_s^*}\,. \tag{17}$$

The rainwater balance, $(8)_3$, involves the precipitation flux $\rho q_{\mathrm{r}} V_{\mathrm{T}}$ with the terminal, quasi-steady relative falling velocity of rain drops in the surrounding air,

$$V_{\mathrm{T}} = V_{\mathrm{T}}^* \,\frac{(\rho q_{\mathrm{r}})^{1+\beta^*}}{\rho^{1/2}}\,. \tag{18}$$

where β^* is a small constant.

This completes the description of the governing equations for the present paper, except for estimates of the various parameterization constants. This, together with asymptotic scaling arguments, will be the theme of Sect. 3.

3 Moist aero-thermodynamics from an asymptotic perspective

3.1 Preliminaries

It is argued in [14,15,17] that the most general approach to analyzing a system with multiple small (or large) parameters involves exploring families of *distinguished limits*. In a distinguished limit, one asymptotic expansion parameter, say $\varepsilon \ll 1$, is introduced, and all other small or large parameters of the system are considered functions of it in the limit as $\varepsilon \to 0$. In combination with techniques of multiple-scale asymptotics, it has been shown in the cited references that a large number of well-established simplified fluid-dynamical models of theoretical meteorology may be recovered in a mathematically unified way on the basis of a single such distinguished limit for the Mach, barotropic Froude, and bulk microscale Rossby numbers from (5), that is,

$$\mathrm{M} \sim \overline{\mathrm{Fr}} \sim \varepsilon^2\,, \quad \mathrm{Ro_B} \sim \varepsilon^{-1} \qquad \text{as} \quad \varepsilon \to 0\,. \tag{19}$$

This approach has proven to be particularly useful in the development of systematic multi-scale models for the tropics in [17] (see also [2,18] for spectacular further developments). Typical values of ε in actual meteorological applications are [14,17],

$$\varepsilon \sim \frac{1}{8} \cdots \frac{1}{6}. \tag{20}$$

Thus far, moist processes have not been modeled explicitly in these analyses. Instead, effective distributions of the induced energy and buoyancy source terms have been assumed, leaving a description of the feedback between moist thermodynamics and fluid mechanics for future work. Here we proceed in this direction by including the additional nondimensional parameters arising in the moisture transport equations from the last section in the distinguished limit.

A key step in making the analysis tractable will be the so-called Newtonian limit for the isentropic exponent, which we borrow from combustion theory [22], (however, see also Bannon's discussion in [1] of the Lipps and Hemler anelastic model [16]),

$$\frac{\gamma - 1}{\gamma} = \Gamma^{**}\varepsilon \;\text{ with }\; \Gamma^{**} = O(1) \quad \text{as}\; \varepsilon \to 0\,. \tag{21}$$

As we shall see below, this limit will allow us to simplify considerably the asymptotic treatment of the stratification of saturated air. With the typical values for ε from (20), and $\gamma \approx 1.4$, we have $(\gamma - 1)/\gamma \approx 2/7 \sim 2\varepsilon$ and the Newtonian limit is compatible with the actual numbers.

For the subsequent analysis, we need plausible asymptotic scalings of the various free parameters in the bulk microphysics model described in the previous section as $\varepsilon \to 0$. For the saturation vapor pressure and saturation mixing ratio we use the data suggested by Emanuel [6]. For the rate coefficients – except for C_{d}^{*} – and other free parameters in $(10)_{3,4}$ we extract estimates from various publications on bulk microphysics parameterizations, [8–10, 12, 25].

The condensation rate coefficient C_{d}^{*} will be assumed to be sufficiently large to allow for an asymptotic derivation of the classical separation of closure schemes into the regimes of saturated and undersaturated air. Besides this split of regimes, the analysis will yield explicit expressions for the (small) super- and subsaturation in nearly saturated air, and it will reveal some interesting analogies with combustion theory.

In the sequel, a $(\cdot)^{**}$ superscript denotes scalar constants that are of order $O(1)$ as $\varepsilon \to 0$.

3.2 Distinguished limits for moist aero-thermodynamics

In this section, we introduce explicit asymptotic coupled limits for all the constant parameters in the bulk microphysics closure scheme. The chosen limits reflect the essence of our literature search for typical orders of magnitude of these parameters. We are quite sure that our choice is compatible with many closure schemes used in practice. However, we do not exclude the possibility that different choices for the coupled limits are also compatible, and that they may lead to different limit regimes. An exploration of the related degrees of freedom is beyond the scope of the present study.

3.2.1 Scalings for the saturation mixing ratio

We employ Bolton's formula (15), for the saturation vapor pressure, which we repeat here for convenience in its dimensional form as given by Emanuel [6],

$$e_s' = e_s^{*\prime} \ \exp\left(A_0^{*} \ \frac{T' - T_0'}{T_1' + (T' - T_0')} \right). \tag{22}$$

Here, primes denote dimensional variables, and

$$e_s^{*\prime} = 611 \ \mathrm{Pa}\,, \quad T_0' = 273.16 \ \mathrm{K}\,, \quad T_1' = 243.5 \ \mathrm{K}\,, \quad A_0^{*} = 17.67\,. \tag{23}$$

By choice of the reference temperature T_0', this formula is gauged to be used relative to typical mid-latitude situations. For the tropics, a more realistic choice of a reference temperature would be $T_0'' = 300$ K, and we use the following identity to provide a rescaled version of the formula in (22),

$$\frac{T' - T_0'}{T_1' + (T' - T_0')} = \frac{T_0'' - T_0'}{T_1''} + \frac{T_1'}{T_1''} \ \frac{T' - T_0''}{T_1'' + (T' - T_0'')}\,, \tag{24}$$

where

$$T_1'' = T_1' + (T_0'' - T_0')\,. \tag{25}$$

With these relations, we may rewrite Bolton's formula with modified coefficients, whereby the first term on the right-hand side in (24) will yield a rescaled value for the pre-exponential, e_s^{*}, and the first factor of the second term will produce a rescaled value for the exponential sensitivity, A^{*}. In fact, we find

$$e_s' = e_s^{*\prime\prime} \ \exp\left(A^{*} \ \frac{T' - T_0''}{T_1'' + (T' - T_0'')} \right), \tag{26}$$

where now

$$e_s^{*\prime\prime} = 3500 \ \mathrm{Pa}\,, \quad T_0'' = 300 \ \mathrm{K}\,, \quad T_1'' = 270 \ \mathrm{K}\,, \quad A^{*} = 16\,. \tag{27}$$

Using the definition of the saturation mixing ratio from (13), we have, in nondimensional terms using $T_{\mathrm{ref}} = T_0''$,

$$q_{\mathrm{vs}}(\theta, p) = \frac{1}{p} \exp\left(A^{*} \ \frac{T(\theta, p) - 1}{T_1^{*} + (T(\theta, p) - 1)} \right). \tag{28}$$

Next, we adopt distinguished limits for the three parameters $q^*_{\rm vs}$, from (17), and A^*, T^*_1 in terms of the expansion parameter ε. With typical values for $\varepsilon \sim 1/8 \dots 1/6$, the following choices appear justified

$$q^*_{\rm vs} = 0.021 \sim \varepsilon^2\, q^{**}_{\rm vs}\,, \qquad A^* = 16 \sim \varepsilon^{-1}\, A^{**}\,, \quad T^*_1 = 0.9 \sim 1 - \varepsilon\, T^{**(1)}_1\,. \tag{29}$$

Here and below, superscripts $^{(i)}$ indicate the ith term in an asymptotic expansion in terms of powers of ε, see also Sect. 1.1. The choices in (29) yield the final formulation of the saturation mixing ratio appropriate for the purposes of asymptotic analysis,

$$q_{\rm vs}(\theta, p) = \frac{1}{p}\ \exp\left(\frac{A^{**}}{\varepsilon}\ \frac{T(\theta, p) - 1}{1 + \left(T(\theta, p) - 1 - \varepsilon T^{**(1)}_1\right)}\right). \tag{30}$$

Notice that we have taken the liberty of simplifying the expression for $q_{\rm vs}$ slightly by neglecting the term $-e_s$ in the denominator of (13). This induces an error of order $O(\varepsilon^2)$ according to the above distinguished limits and could be easily accounted for if needed, but the correction will not change the leading-order results to be derived below.

3.2.2 Scalings of mixing ratios and moisture conversion rates

From (30) we conclude that the saturation mixing ratio, in absolute terms, scales as

$$q^*_{\rm vs} = O(\varepsilon^2) \qquad \text{as } \varepsilon \to 0\,. \tag{31}$$

Typical water vapor, cloud water, and precipitation mixing ratios cannot be larger than the water vapor saturation level, because the latter represents the limiting water supply, and the atmosphere cannot store large amounts of liquid water. As a consequence, we have let

$$q_{\rm v_{ref}} = q_{\rm c_{ref}} = q_{\rm r_{ref}} = q_{\rm vs_{ref}} = q^*_{\rm vs} = \varepsilon^2 q^{**}_{\rm vs} \tag{32}$$

in scaling the moisture variables in (7). Using these scalings, we have estimated plausible distinguished limits for the rate coefficients in (10) for autoconversion of cloud water to precipitation, $C^*_{\rm ac}$, for the collection of cloud water by the falling precipitation, $C^*_{\rm cr}$, and for evaporation, $C^*_{\rm ev}$, from [8,9,25]. The resulting asymptotic scalings are

$$C^*_{\rm ac} \equiv C^{**}_{\rm ac}\,, \qquad C^*_{\rm cr}\, q_{\rm c_{ref}} = \frac{1}{\varepsilon}\, C^{**}_{\rm cr}\,, \qquad C^*_{\rm ev}\, q^{1/2}_{\rm r_{ref}} = C^{**}_{\rm ev}\,. \tag{33}$$

Below we will derive the asymptotic consequences of very rapid condensation/evaporation of cloud water. To this end, we require the condensation source term for characteristic values of $q_{\rm v}$, $q_{\rm c}$ to be very large when nondimensionalized by our reference time scale $t_{\rm ref} \sim 1,000$ s. This implies

$$C^*_{\rm d}\, q_{\rm c_{ref}} \gg 1\,, \tag{34}$$

and accordingly we let

$$C^*_{\rm d} q^*_{\rm vs} = \frac{1}{\varepsilon^n} C^{**}_{\rm d} \qquad \text{with} \qquad n \gg 1\,. \tag{35}$$

The closure scheme for autoconversion of cloud water to precipitation involves a lower autoconversion threshold, $q^*_{\rm c} \approx 5 \times 10^{-4}$, for the cloud water mixing ratio. This is compatible with a scaled threshold value

$$q^*_{\rm c} = \varepsilon^3\, q^{**}_{\rm c}\,. \tag{36}$$

Cloud-resolving simulations generally show characteristic values of cloud water content to be an order of magnitude smaller than that of water vapor and precipitation. This would justify an $O(\varepsilon^3)$ scaling of cloud water content right from the outset, instead of adopting a common reference value for all water components as in (32). However, such a scaling will be the *result* of our analysis, rather than being assumed.

For the nucleation constant, q^*_{cn}, we assume the same scaling as for $q^*_{\rm c}$, i.e.,

$$q^*_{cn} = \varepsilon^3\, q^{**}_{cn}\,. \tag{37}$$

As pointed out to the authors by B. Stevens, it is quite likely that the asymptotic scalings for $q^*_{\rm c}$ and q^*_{cn} in (36) and (37), respectively, are not uniformly valid. For example, considerably different values for these parameters may be expected for maritime and continental clouds. Exploration of these degrees of freedom may shed some systematic light various interesting microphysical effects in bulk cloud dynamics.

3.2.3 Scalings for latent heat and its effect on potential temperature

The dimensionless latent heat per unit mass of water vapor, L^*, from (9) may be estimated by $L^* \approx 30$. Compatible scalings would read $L^* \sim \varepsilon^{-1}$ or $L^* \sim \varepsilon^{-2}$. However, there is a constraint from thermodynamics: The Clausius–Clapeyron relation for the saturation vapor pressure states that $\frac{T}{e_s}\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{L}{RT}$. Bolton's formula, used above in (26), is a close approximation to the integral of the Clausius–Clapeyron equation and it follows that $L^* = \frac{L}{(RT)_{\mathrm{ref}}} \sim A^*$. Thus, consistency with both Clausius–Clapeyron *and* the earlier scaling $A^* = \varepsilon^{-1}A^{**}$ implies

$$L^* = \frac{1}{\varepsilon} L^{**} \, . \tag{38}$$

To assess the order of magnitude of latent-heat-induced variations of potential temperature we combine (9), $(8)_1$, and (32) to obtain

$$\delta\theta \sim \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{\varepsilon}L^{**}\right)\left(\varepsilon^2 q_{\mathrm{vs}}^{**}\delta q_{\mathrm{v}}\right) \, . \tag{39}$$

Depending on whether or not we employ the Newtonian limit from (21), we find different scalings for the variation of potential temperature, namely

$$\delta\theta = \left\{\begin{array}{lll} O(\varepsilon) & \text{for} & \dfrac{\gamma-1}{\gamma} = O(1) \\[2ex] O(\varepsilon^2) & \text{for} & \dfrac{\gamma-1}{\gamma} = \varepsilon\,\Gamma^{**}\,, \quad \Gamma^{**} = O(1) \end{array}\right\} \quad \text{as } \varepsilon \to 0 \, . \tag{40}$$

We do adopt the Newtonian limit here, i.e., the second option, and conclude from (9) that the latent-heat-induced potential temperature source term should scale as

$$S_\theta^q = \varepsilon^2\,\Gamma^{**}L^{**}q_{\mathrm{vs}}^{**}\,\frac{\theta}{p}\,(C_{\mathrm{d}} - C_{\mathrm{ev}}) \, . \tag{41}$$

3.3 Asymptotically scaled governing equations

Here we summarize the nondimensional governing equations as they appear when the distinguished limits from the last section are introduced.

Mass, momentum, energy

$$\begin{aligned} \rho_t + \nabla_\parallel \cdot (\rho\boldsymbol{u}) + (\rho w)_z &= 0 \\ \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla_\parallel\boldsymbol{u} + w\boldsymbol{u}_z + \varepsilon\, f\,(\boldsymbol{\Omega}\times\boldsymbol{v})_\parallel + \frac{1}{\varepsilon^4}\nabla_\parallel p &= \boldsymbol{D}_{\boldsymbol{u}} \\ w_t + \boldsymbol{u}\cdot\nabla_\parallel w + w w_z + \varepsilon\, f\,(\boldsymbol{\Omega}\times\boldsymbol{v})_\perp + \frac{1}{\varepsilon^4}\,p_z &= D_w - \frac{1}{\varepsilon^4} \\ \theta_t + \boldsymbol{u}\cdot\nabla_\parallel\theta + w\theta_z &= D_\theta + \varepsilon^2\left(\tilde{S}_\theta^{\,\varepsilon} + S_\theta^{q,\varepsilon}\right) \end{aligned} \tag{42}$$

where

$$S_\theta^{q,\varepsilon} = \Gamma^{**}L^{**}\,q_{\mathrm{vs}}^{**}\,\frac{\theta}{p}\left[\frac{1}{\varepsilon^n}\hat{C}_{\mathrm{d}} - \hat{C}_{\mathrm{ev}}\right] . \tag{43}$$

The scaled condensation and evaporation terms, $\hat{C}_{\mathrm{d}}$, $\hat{C}_{\mathrm{ev}}$ will be defined shortly, while we leave the non-moisture-related diabatic term, $\tilde{S}_\theta^{\,\varepsilon}$, as an externally given source term here and below. The only assumption, to be explained later, will be the order-of-magnitude estimate $\tilde{S}_\theta^{\,\varepsilon} = O(\varepsilon)$ as $\varepsilon \to 0$.

For the scaled moisture balances we recall from (8) that

$$
\begin{aligned}
q_{\mathrm{v}\,,t} + \boldsymbol{u}\cdot\nabla_{\|} q_{\mathrm{v}} + w\; q_{\mathrm{v}\,,z} &= -\frac{1}{\varepsilon^n}\hat{C}_{\mathrm{d}} + \hat{C}_{\mathrm{ev}} + D_{q_{\mathrm{v}}}\\
q_{\mathrm{c}\,,t} + \boldsymbol{u}\cdot\nabla_{\|} q_{\mathrm{c}} + w\; q_{\mathrm{c}\,,z} &= \frac{1}{\varepsilon^n}\hat{C}_{\mathrm{d}} - \frac{1}{\varepsilon}\hat{C}_{\mathrm{cr}} - \hat{C}_{\mathrm{ac}} + D_{q_{\mathrm{c}}}\\
q_{\mathrm{r}\,,t} + \boldsymbol{u}\cdot\nabla_{\|} q_{\mathrm{r}} + w\; q_{\mathrm{r}\,,z} + \frac{1}{\rho}(V_{\mathrm{T}}^{**}\rho q_{\mathrm{r}})_z &= \frac{1}{\varepsilon}\hat{C}_{\mathrm{cr}} - \hat{C}_{\mathrm{ev}} + \hat{C}_{\mathrm{ac}} + D_{q_{\mathrm{r}}}\,,
\end{aligned}
\tag{44}
$$

with the source terms defined by

$$
\begin{aligned}
\hat{C}_{\mathrm{d}} &= C_{\mathrm{d}}^{**}\,(q_{\mathrm{v}} - q_{\mathrm{vs}})\; H_{\circ}(q_{\mathrm{c}}, q_{\mathrm{v}}, q_{\mathrm{vs}})(q_{\mathrm{c}} + \varepsilon q_{cn}^{**})\\
\hat{C}_{\mathrm{ev}} &= -C_{\mathrm{ev}}^{**}\,(q_{\mathrm{v}} - q_{\mathrm{vs}})\; H_{>}(q_{\mathrm{r}}) q_{\mathrm{r}}^{\frac{1}{2}}\\
\hat{C}_{\mathrm{cr}} &= C_{\mathrm{cr}}^{**}\, q_{\mathrm{c}} q_{\mathrm{r}}\\
\hat{C}_{\mathrm{ac}} &= C_{\mathrm{ac}}^{**}\,\max\left(0, q_{\mathrm{c}} - \varepsilon\, q_{\mathrm{c}}^{**}\right)\,.
\end{aligned}
\tag{45}
$$

The scaled saturation vapor mixing ratio reads

$$
q_{\mathrm{vs}}(\theta, p) = \frac{1}{p}\,\exp\left(\frac{A^{**}}{\varepsilon}\,\frac{T(\theta,p) - 1}{1 + \left(T(\theta,p) - 1 - \varepsilon T_1^{**(1)}\right)}\right), \quad \text{with} \quad T(\theta,p) = \theta\, p^{\varepsilon\Gamma^{**}}\,. \tag{46}
$$

Finally, the scaled rain flux for both regimes becomes

$$
V_{\mathrm{T}} = V_{\mathrm{T}}^{**}\,\rho^{1/2} q_{\mathrm{r}}\,. \tag{47}
$$

In (43)–(47) all dependencies of the equations on ε are *explicit*, i.e., any term that does not exhibit a dependence on ε is of order $O(1)$ asymptotically as $\varepsilon \to 0$. Also, for simplicity of exposition, we have neglected small deviations of various exponents from integer values or rational numbers, assuming $\alpha^*, \beta^*, \delta^* = 0$ in (10), (18).

In the remainder of this section we construct first asymptotic results from these asymptotically scaled governing equations. In particular, we will consider the hydrostatic background state and the asymptotics of the saturation mixing ratio in some detail.

3.4 Background stratification in the Newtonian limit

3.4.1 Leading-order pressure and density distributions

Order-of-magnitude estimates in [17], based on typical buoyancy frequencies, suggest that variations of potential temperature, θ, throughout the troposphere are small. The actual numbers are a bit ambiguous, so that either of the following two alternatives would be reasonable,

$$
\frac{\theta' - T'_{\mathrm{ref}}}{T'_{\mathrm{ref}}} = O(\varepsilon) \qquad \text{or} \qquad \frac{\theta' - T'_{\mathrm{ref}}}{T'_{\mathrm{ref}}} = O(\varepsilon^2) \qquad \text{as} \qquad \varepsilon \to 0\,, \tag{48}
$$

where primes again denote dimensional quantities. Nondimensionally, we may therefore expand the potential temperature $\theta = \theta'/T'_{\mathrm{ref}}$ as

$$
\theta = 1 + \varepsilon\,\theta^{(1)} + \cdots \qquad \text{or} \qquad \theta = 1 + \varepsilon^2\,\theta^{(2)} + \cdots\,. \tag{49}
$$

These alternatives are also compatible with the estimates given in (40) for the effect of latent-heat release on potential temperature. We notice in passing that in the Newtonian Limit a first-order potential temperature perturbation, $\varepsilon\theta^{(1)}$, cannot be associated with the direct effects of latent-heat release (which is of order $O(\varepsilon^2)$). Rather, it would have to be the result, e.g., of the longer-term radiative balance, which is indirectly affected by the moisture distribution. We restrict ourselves here to the analysis of the direct effects of latent heating.

Combining (49) with the Newtonian expansion of the temperature definition in (16),

$$T(\theta, p) = \theta \, \left(1 + \varepsilon \, \Gamma^{**} \ln(p) + O(\varepsilon^2)\right) , \tag{50}$$

we find

$$T(\theta, p) = 1 + \varepsilon \, \left(\theta^{(1)} + \Gamma^{**} \ln(p)\right) + O(\varepsilon^2) \qquad \text{or} \qquad T(\theta, p) = 1 + \varepsilon \, \Gamma^{**} \ln(p) + O(\varepsilon^2) . \tag{51}$$

As usual in meteorological theories, we will pursue asymptotic expansions about a hydrostatic background state below. Therefore, the leading-order pressure $p^{(0)} \equiv p_h(z)$ will satisfy the hydrostatic balance (in terms of nondimensional variables)

$$\frac{\mathrm{d}p_h}{\mathrm{d}z} = -\rho_h = -\frac{p_h^{1/\gamma}}{\overline{\theta}}, \tag{52}$$

with an appropriately averaged potential temperature stratification, $\overline{\theta}(z)$. The exact solution (for $p_h(0) = 1$) is

$$p_h(z) = \left(1 - \frac{\gamma - 1}{\gamma} \int\limits_0^z \overline{\theta}^{-1}(\zeta) \, \mathrm{d}\zeta\right)^{\frac{\gamma}{\gamma-1}} . \tag{53}$$

Immediately we have, for the density and temperature of the background state,

$$\rho_h(z) = \frac{1}{\overline{\theta}(z)} \left(1 - \frac{\gamma - 1}{\gamma} \int\limits_0^z \overline{\theta}^{-1}(\zeta) \, \mathrm{d}\zeta\right)^{\frac{1}{\gamma-1}} , \qquad T_h(z) = \overline{\theta}(z) \left(1 - \frac{\gamma - 1}{\gamma} \int\limits_0^z \overline{\theta}^{-1}(\zeta) \, \mathrm{d}\zeta\right) . \tag{54}$$

Next we consider in the first-order expansion in ε of these expressions under the Newtonian limit from (21) assuming no more than $O(\varepsilon)$ overall variations of θ from (49), i.e.,

$$T_h(z; \varepsilon) = 1 + \varepsilon T^{(1)}(z) + o(\varepsilon) = 1 + \varepsilon \, \left(\overline{\theta}^{(1)}(z) - \Gamma z\right) + o(\varepsilon) , \tag{55}$$

and

$$p_h^{(0)}(z) = \lim_{\varepsilon \to 0} (1 - \varepsilon \Gamma z)^{\frac{1}{\varepsilon \Gamma}} = \exp(-z) , \qquad \rho_h^{(0)}(z) = \exp(-z) . \tag{56}$$

At first order, we find

$$p_h^{(1)}(z) = \Gamma^{**} \left(-\frac{1}{2} z^2\right) p_h^{(0)}(z) , \qquad \rho_h^{(1)}(z) = \Gamma^{**} \left(z - \frac{1}{2} z^2\right) \rho_h^{(0)}(z) . \tag{57}$$

Explicit expressions for higher-order pressures and densities will be worked out below where needed.

3.4.2 Asymptotics of the saturation vapor mixing ratio

Here we demonstrate how the Newtonian limit and the small perturbation assumption for potential temperature discussed in the last section alleviate the following technical multilayer difficulty that arises in the analysis of deep atmospheric flows with vertical extent comparable to the pressure scale height. By definition, within such a flow the thermodynamic pressure varies by order $O(1)$, and so does the temperature in general, according to (16). Following (30), which exhibits explicitly the exponential large-activation-energy sensitivity of the saturation vapor content with respect to temperature, we must conclude that q_{vs} will vary by many order of magnitude in ε as one passes through the atmospheric layer in the vertical direction. Asymptotic expansions in terms of powers of ε will then essentially have to distinguish many vertical layers, separated from each other by a drop of the saturation mixing ratio by factors of powers of ε.

While doable in principle, such multilayer expansions would be extremely tedious and they do not seem to be necessary, considering that the main distinguishable layers – at least in terms of moisture physics – are the planetary boundary layer, the bulk troposphere, a layer around the tropopause, and then the essentially dry stratosphere. In mathematical terms we can avoid this issue by inserting the temperature expansion (51) in the asymptotic representation of the saturation mixing ratio in (46),

$$q_{\mathrm{vs}} = \exp\left(A^{**}\theta^{(1)} - \left[A^{**}\Gamma^{**} - 1\right] z\right) (1 + O(\varepsilon))\,. \tag{58}$$

This asymptotic representation removes the exponential sensitivity of q_{vs} on ε. Notice that (58) yields an explicit expression for the nondimensional moisture scale height

$$h^q_{\mathrm{scale}} = \left[A^{**}\Gamma^{**} - 1\right]^{-1}\,. \tag{59}$$

In the sequel we will adopt the potential temperature scaling from [17], i.e., $\theta = 1 + \varepsilon^2\theta^{(2)} + O(\varepsilon^2)$, so that (58) simplifies to

$$q_{\mathrm{vs}} = q^{(0)}_{\mathrm{vs}}(z) + \varepsilon q^{(1)}_{\mathrm{vs}}(z) + o(\varepsilon) \tag{60}$$

where

$$\begin{aligned} q^{(0)}_{\mathrm{vs}}(z) &= \exp\left(-\left[A^{**}\Gamma^{**} - 1\right] z\right) \\ q^{(1)}_{\mathrm{vs}}(z) &= q^{(0)}_{\mathrm{vs}}(z)\left[\left(A^{**}\Theta_2(z) - \frac{1}{2}\,A^{**}\Gamma^{**2} z^2\right) + \exp(-z)\left(A^{**}\Gamma^{**} - 1\right)\, p^{(1)}_h(z)\right], \end{aligned} \tag{61}$$

with $p^{(1)}_h$ from (57). As one plausibility check for the formula defining $q^{(0)}_{\mathrm{vs}}$, let us assume $\varepsilon = 1/7$ for the moment, so that $A^{**} \approx 2.3$, $\Gamma^{**} = 2$, and $A^{**}\Gamma^{**} - 1 \approx 3.6$. Then the saturation vapor content will appropriately vanish as $z \to \infty$.

3.4.3 Brunt–Väisälä, CAPE, and the background stratification

With the scalings for potential temperature in (49), which are based on typical values of the Brunt–Väisälä frequency, we have seen that the *leading-order* solutions for pressure and density are those of a neutrally stratified atmosphere. Stratification enters at higher order only. At this stage it is undecided whether we should assume the stronger stratification involving $\Theta_1(z)$ as implied by $(49)_1$, or the weaker version with $\Theta_1 \equiv 0$ and $\theta = 1 + \varepsilon^2\Theta_2(z) + \cdots$ as implied by $(49)_2$. The latter scaling was *assumed* in our earlier study in [17].

Here we proceed one step further by comparing the background stratification with a moist adiabatic one: According to Emanuel [6] typical values of the convectively available potential energy (CAPE) are $\mathrm{CAPE}' \sim 25, \ldots, 400\ \mathrm{m}^2/\mathrm{s}^2$. In nondimensional terms this means

$$\mathrm{CAPE} = \frac{\mathrm{CAPE}'}{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}} = 6 \times 10^{-4}, \ldots, 5 \times 10^{-3}\,. \tag{62}$$

With $\varepsilon = 1/8, \ldots, 1/6$ the best match of these values to powers of ε in the sense of a distinguished limit reads

$$\mathrm{CAPE} = O(\varepsilon^4) \ldots O(\varepsilon^3) \qquad \text{as } \varepsilon \to 0\,. \tag{63}$$

In the context of (40) we have shown that, if we adopt the Newtonian Limit for the isentropic exponent, *and* we assume that the background stratification of potential temperature is affected nontrivially by moist processes, then we are forced to assume the weaker stratification for θ in $(49)_2$. Thus, variations of the potential temperature are of the order $\delta\theta = O(\varepsilon^2)$ as $\varepsilon \to 0$. The estimates for typical values of CAPE in (63) imply furthermore that deviations of the background stratification from a moist adiabat must be at least another order of magnitude smaller, so that $\Theta_2(z)$ must satisfy the moist adiabatic equation

$$\frac{\mathrm{d}\Theta_2}{\mathrm{d}z} = -\frac{\Gamma^{**}L^{**}q^{**}_{\mathrm{vs}}}{p_0(z)}\frac{\mathrm{d}q^{(0)}_{\mathrm{vs}}}{\mathrm{d}z} = \Gamma^{**}L^{**}q^{**}_{\mathrm{vs}}\left[A^{**}\Gamma^{**} - 1\right]\exp\left(-\left[A^{**}\Gamma^{**} - 2\right] z\right). \tag{64}$$

The reader may confer (40) to verify the scalings in terms of powers of ε. The last equality follows from the explicit representation of the saturation water content in (60).

Equation (64) will hold within the vertical layer within which moist convection can occur. If there is a more rapid increase of Θ_2 above some height z^* then this shuts off convection above this level, and limits further accumulation of CAPE. Here we are interested in deep convection processes and assume that (64) holds throughout the troposphere and the exact, explicit solution for $\Theta_2(z)$ then reads

$$\Theta_2(z) = \Theta_2(0) + \Gamma^{**} L^{**} q_{\mathrm{vs}}^{**} \, \frac{A^{**}\Gamma^{**} - 1}{A^{**}\Gamma^{**} - 2} \, \left(1 - \exp\left(-\left[A^{**}\Gamma^{**} - 2\right] z\right)\right) . \tag{65}$$

Thus we have determined the background stratification up to order $O(\varepsilon^2)$ explicitly in terms of moist thermodynamics, except for the base value $\Theta_2(0)$. This last degree of freedom allows us to adjust the thermodynamic conditions at sea level if these do not match with the chosen reference state, p_{ref}, ρ_{ref}, in an actual application.

This completes the preparatory scaling analysis of the compressible flow equations with bulk microphysics closure. We have identified small or large nondimensional parameters and introduced suitable distinguished limits to couple them to an asymptotic expansion parameter, $\varepsilon \ll 1$. In the next two sections we move on to derive two multiscale models – more precisely two single-time- multiple-space-scale models – via systematic asymptotic multiple-scale expansions.

4 Short-time evolution of bulk microscale hot towers in a convective-scale environment

4.1 Scalings, asymptotic ansatz, and key results

In the present section we summarize the main results of the asymptotic analysis to provide a compact overview. Details of the derivations are given in subsequent sections.

Here we consider the unsteady evolution of deep convective *hot towers* with characteristic horizontal scale $\ell_\mu \sim 1$ km embedded in a convective-scale environment with characteristic length comparable to the pressure scale height, $\ell_{\mathrm{ref}} = h_{\mathrm{scale}} \sim 10$ km. We restrict to a simplified setting involving a vertically sheared horizontal flow with embedded small scale vertical towers. The latter may be interpreted as columns in the sense of typical column model parameterizations of deep convection (see e.g., [7]).

We are interested in the short-time evolution of such convective columns including their onset and/or decay and their deformation by the vertically sheared horizontal motions. To this end we consider typical time scales associated with horizontal advection across a characteristic diameter of the columns. Focusing on deep convection, we assume vertical variation on scales comparable to the pressure scale height. These considerations induce the multiple-scale asymptotic ansatz,

$$\mathbf{U}(\boldsymbol{x}, z, t; \varepsilon) = \sum_i \varepsilon^i \mathbf{U}^{(i)}(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) , \qquad \text{where} \qquad \boldsymbol{\eta} = \frac{\boldsymbol{x}}{\varepsilon} , \tau = \frac{t}{\varepsilon} . \tag{66}$$

Here $\mathbf{U}$ represents the tuple of dependent variables.

To reflect the assumed simplified structure of the velocity field with a *convective-scale* horizontal background flow, but small-scale columnar vertical motion, we let

$$\boldsymbol{u} = \boldsymbol{u}^{(0)}(\boldsymbol{x}, z, \tau) + O(\varepsilon) \qquad \text{but } w = w^{(0)}(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) + O(\varepsilon) . \tag{67}$$

At the same time we adopt the scaling for the background potential temperature stratification as employed in [17] and discussed above, i.e.,

$$\theta = 1 + \varepsilon^2 \, \Theta_2(z) + \varepsilon^3 \, \theta^{(3)}(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) + O(\varepsilon^4) . \tag{68}$$

The horizontally homogeneous stratification from (68), via hydrostatic balance at the second order, will imply the pressure and density expansions

$$(p, \rho) = (p_0, \rho_0)(z) + \varepsilon \, (P_1, R_1)(z) + \varepsilon^2 \, (P_2, R_2)(z) + \varepsilon^3 \, (p^{(3)}, \rho^{(3)})(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) + O(\varepsilon^4) . \tag{69}$$

We have used here the Newtonian Limit for the isentropic exponent, $(\gamma - 1)/\gamma = \varepsilon \, \Gamma^{**}$, assumed the latent heat per unit mass and the Arrhenius' activation energy for the saturation vapor pressure to be large, so that $L/(p_{\mathrm{ref}}/\rho_{\mathrm{ref}}) = L^{**}/\varepsilon$, and $A^* = A^{**}/\varepsilon$, and taken into account that the reference value, $q_{\mathrm{vs}}^* = \varepsilon^2 q_{\mathrm{vs}}^{**}$, for the water vapor, cloud water, and precipitation mixing ratios is small of order $O(\varepsilon^2)$. The mixing ratios *scaled by this reference value* are expanded as

$$(q_{\mathrm{v}}, q_{\mathrm{r}}) = \left(q_{\mathrm{v}}^{(0)}, q_{\mathrm{c}}^{(0)}, q_{\mathrm{r}}^{(0)}\right)(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) + \varepsilon \, \left(q_{\mathrm{v}}^{(1)}, q_{\mathrm{c}}^{(1)}, q_{\mathrm{r}}^{(1)}\right)(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) + o(\varepsilon^2) . \tag{70}$$

Multiple-scale analysis (see the next section) then yields the following leading-order system of coupled equations for the evolution of the convective-scale background and the bulk microscale columns.

Linearized convective-scale momentum balance

$$
\begin{aligned}
\boldsymbol{u}_{\tau}^{(0)} + \nabla_{\boldsymbol{x}}\pi^{(3)} &= 0 \\
\overline{w^{(0)}}_{\tau} + \pi_z^{(3)} &= \overline{\theta^{(3)}} \\
\overline{\theta^{(3)}}_{\tau} + \overline{w^{(0)}}\frac{\mathrm{d}\Theta_2}{\mathrm{d}z} &= \frac{\Gamma^{**}L^{**}}{p_0}\overline{C^{(0)}} \\
\rho_0\nabla_{\boldsymbol{x}}\boldsymbol{u}^{(0)} + \left(\rho_0\overline{w^{(0)}}\right)_z &= 0,
\end{aligned}
\tag{71}
$$

where $\pi^{(3)} = p^{(3)}/\rho_0$, an overbar denotes averaging over the fast spacial coordinate, $\boldsymbol{\eta}$, and $C^{(0)}$ will be defined shortly.

Bulk microscale column dynamics On the small scale we find that

$$
\begin{aligned}
\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}}\right)\widetilde{w^{(0)}} &= \widetilde{\theta^{(3)}} \\
\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}}\right)\widetilde{\theta^{(3)}} + \widetilde{w^{(0)}}\frac{\mathrm{d}\Theta_2}{\mathrm{d}z} &= \frac{\Gamma^{**}L^{**}}{p_0}\widetilde{C^{(0)}}.
\end{aligned}
\tag{72}
$$

where, for any variable ϕ, we let $\widetilde{\phi} = \phi - \overline{\phi}$.

Through the vertical velocity, $w^{(0)}$, and the condensation–evaporation source term, $C^{(0)}$, these equations couple with the moisture transport equations. For $C^{(0)}$ we must distinguish the two regimes of saturated and undersaturated air. In saturated air, only condensation–evaporation of cloud water is possible, i.e., $C^{(0)} = C_{\mathrm{d}}^{(0)}$, whereas in undersaturated air we have only evaporation of precipitation, i.e., $C^{(0)} = C_{\mathrm{ev}}^{(0)}$. The water vapor content acts as an indicator distinguishing the two regimes, so that

$$
C^{(0)} = H_{q_{\mathrm{v}}}C_{\mathrm{d}}^{(0)} + \left[1 - H_{q_{\mathrm{v}}}\right]C_{\mathrm{ev}}^{(0)} \quad \text{where} \quad H_{q_{\mathrm{v}}} = H_{\geq}(q_{\mathrm{v}}^{(0)} - q_{\mathrm{vs}}^{(0)})
\tag{73}
$$

with $H_{\geq}(\cdot)$ from (11).

Moisture transport equations for saturated air ($H_{q_{\mathrm{v}}} = 1$) The partial source terms $C_{\mathrm{d}}^{(0)}$ and $C_{\mathrm{ev}}^{(0)}$ are defined through the following two equation sets.

$$
\begin{aligned}
C_{\mathrm{d}}^{(0)} = C_{\mathrm{d}}^{**}\,\delta q_{\mathrm{v}}^{(n^*)}\,(q_{\mathrm{c}}^{(1)} + q_{cn}^{**}) &= -\left[\left(\widetilde{w^{(0)}} + \overline{w^{(0)}}\right)\frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} - D_{q_{\mathrm{v}}}^{(0)}\right] \\
\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}}\right)q_{\mathrm{c}}^{(1)} &= H_{\geq}(q_{\mathrm{c}}^{(1)})\,C_{\mathrm{d}}^{(0)} - C_{\mathrm{cr}}^{**}q_{\mathrm{r}}^{(0)}\,q_{\mathrm{c}}^{(1)} \\
\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}}\right)q_{\mathrm{r}}^{(0)} &= 0
\end{aligned}
\tag{74}
$$

and the

Moisture transport equations for undersaturated air ($H_{q_{\mathrm{v}}} = 0$)

$$
\begin{aligned}
C_{\mathrm{ev}}^{(0)} &= C_{\mathrm{ev}}^{**}\left(q_{\mathrm{vs}}^{(0)}(z) - q_{\mathrm{v}}^{(0)}\right)q_{\mathrm{r}}^{(0)\,1/2} \\
\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}}\right)q_{\mathrm{v}}^{(0)} &= 0 \\
\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla_{\boldsymbol{\eta}}\right)q_{\mathrm{r}}^{(0)} &= 0
\end{aligned}
\tag{75}
$$

respectively.

In both the saturated and undersaturated air regimes we find the leading-order water vapor and precipitation mixing ratios, $q_{\mathrm{v}}^{(0)}, q_{\mathrm{r}}^{(0)}$, to be frozen on the considered time scale while being advected by the background flow. In undersaturated air, there is persistent evaporative cooling which will, for any $\theta^{(3)}$ initial data, ultimately induce vertical downdrafts through the small-scale momentum balance in $(72)_1$. At the same time, this persistent cooling will, by contributing to the average source term $\overline{C^{(0)}}$, also reduce the mesoscale mean potential temperature as seen in (71), thereby inducing mesoscale mean downdrafts.

Interestingly, changes of the small-scale potential temperature fluctuation, $\widetilde{\theta^{(3)}}$, are driven by the *fluctuation*, $\widetilde{C^{(0)}} = C^{(0)} - \overline{C^{(0)}}$, of the condensation–evaporation source term. As a consequence, even if there was no latent-heat release from condensation at all, we would still see a positive source term for $\widetilde{\theta^{(3)}}$ in regions of saturated air, and these would tend to induce small-scale updrafts via $(72)_2$. Of course, through (74) such updrafts can ultimately produce latent-heat release, and thus provide a positive feedback, thereby amplifying themselves.

The condensation source term in (74) has an interesting structure. The only moisture variable it involves explicitly is the given background stratification of the saturation vapor mixing ratio, $q_{\mathrm{vs}}^{(0)}(z)$. Thus, wherever $H_{\geq}(q_{\mathrm{c}}^{(1)})$ is positive, the flow dynamics governing $w^{(0)}$ entirely controls the direct condensation rate through the first term in the square bracket. Notice, however, the (horizontal) turbulent transport term, $D_{q_{\mathrm{v}}}^{(0)}$, i.e., the second term in the bracket. The entire square bracket is the *effective* condensation rate, rather than only the obvious first term. This is referred to in the literature as the *implicit definition of condensation* [8], and here it is a direct result of the asymptotic analysis. Due to the columnar structure of the flow fields considered here, turbulent transport will be dominantly horizontal, with a typical closure reading $D_{q_{\mathrm{v}}}^{(0)} = \nabla_{\boldsymbol{\eta}} \cdot (K \nabla_{\boldsymbol{\eta}} q_{\mathrm{v}}^{(0)})$. Within regions of saturated air, $q_{\mathrm{v}}^{(0)} \equiv q_{\mathrm{vs}}^{(0)}(z)$ and the transport term will vanish. At the *edges* of saturated regions, there will, however, be an abrupt change with possible jumps in the gradient of $q_{\mathrm{v}}^{(0)}$, leading to very strong horizontal transport. This is a situation familiar from thin premixed flames in combustion, (see, e.g., [24]), which are defined by a local quasi-stationary advection–reaction–diffusion balance. A detailed exploration of the structure of these saturation–undersaturation boundaries is, however, beyond the scope of the present paper.

4.2 Key steps of the derivations

Here we collect those first few leading terms from each of the governing equations that are relevant for deriving the simplified asymptotic equations announced in the previous section.

Horizontal momentum:

$$\begin{aligned} \nabla_{\boldsymbol{\eta}} p^{(3)} &= 0 \\ \rho_0 \boldsymbol{u}_{\tau}^{(0)} + \nabla_{\boldsymbol{\eta}} p^{(4)} + \nabla_{\boldsymbol{x}} p^{(3)} &= \rho_0\, \boldsymbol{D}_{\boldsymbol{u}}^{(-1)} \end{aligned} \tag{76}$$

Vertical momentum:

$$\begin{aligned} \frac{\mathrm{d}p_0}{\mathrm{d}z} &= -\rho_0 \\ \frac{\mathrm{d}P_1}{\mathrm{d}z} &= -R_1 \\ \frac{\mathrm{d}P_2}{\mathrm{d}z} &= -R_2 \\ \rho_0 \frac{D^{(0)}}{D\tau} w^{(0)} + p_z^{(3)} &= -\rho^{(3)} + \rho_0\, D_w^{(-1)} \end{aligned} \tag{77}$$

where

$$\frac{D^{(0)}}{D\tau} = \left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) . \tag{78}$$

Mass:

$$\rho_0 \left(\nabla_{\boldsymbol{\eta}} \cdot u^{(1)} + \nabla_{\boldsymbol{x}} \cdot u^{(0)}\right) + \left(\rho_0 w^{(0)}\right)_z = 0 \tag{79}$$

Potential temperature:

$$\frac{D^{(0)}}{D\tau}\theta^{(3)} + w^{(0)}\frac{\mathrm{d}\Theta_2}{\mathrm{d}z} = \frac{\Gamma^{**}L^{**}q_{\mathrm{vs}}^{**}}{p_0}\left(C_{\mathrm{d}}^{(0)} - C_{\mathrm{ev}}^{(0)}\right) + D_\theta^{(2)} . \tag{80}$$

The evaporation and condensation source terms $C_{\mathrm{ev}}^{(0)}$, $C_{\mathrm{d}}^{(0)}$ will be specified below for the two separate situations of nearly saturated and undersaturated air, respectively.

4.2.1 The dry air and saturated air regimes

In the water vapor and cloud water transport equations from (45) the condensation–evaporation term dominates. According to $(10)_1$, (35) the condensation rate reads $C_{\mathrm{d}} = \varepsilon^{-n} C_{\mathrm{d}}^{**}(q_{\mathrm{v}} - q_{\mathrm{vs}}) H_\circ(\dots)(q_{\mathrm{c}} + q_{\mathrm{c}\,n}^{*})$. We expand it as

$$C_{\mathrm{d}} = \frac{1}{\varepsilon^n} C_{\mathrm{d}}^{(-n)} + \frac{1}{\varepsilon^{n-1}} C_{\mathrm{d}}^{(-n+1)} + \cdots . \tag{81}$$

Then the leading terms in the expansion of $(45)_1$ yield

$$C_{\mathrm{d}}^{(i)} = 0 \quad \text{for} \quad i = -n, \dots, -1 . \tag{82}$$

Denoting the distance from saturation by δq_{v}, such that,

$$q_{\mathrm{v}} - q_{\mathrm{vs}} = \delta q_{\mathrm{v}} = \delta q_{\mathrm{v}}^{(0)} + \varepsilon\, \delta q_{\mathrm{v}}^{(1)} + \cdots \tag{83}$$

we may rewrite the first result in (82) as

$$C_{\mathrm{d}}^{(-n)} = C_{\mathrm{d}}^{**}\, \delta q_{\mathrm{v}}^{(0)}\, H_\circ(q_{\mathrm{c}}, q_{\mathrm{v}}, q_{\mathrm{vs}}) q_{\mathrm{c}}^{(0)} = 0 . \tag{84}$$

This and analogous expansions for the higher-order terms in (82) lead to the following alternatives:

Nearly saturated air

$$\delta q_{\mathrm{v}}^{(i)} = 0 \quad \text{for } i = 0, \dots, n-1 , \tag{85}$$

and

Undersaturated air

$$H_>(q_{\mathrm{c}}) = 0 \quad \text{i.e. } q_{\mathrm{c}} \equiv 0 . \tag{86}$$

To verify the last statement, we recall that undersaturation means $q_{\mathrm{v}} < q_{\mathrm{vs}}$ and take into account the definition of $H_\circ$ in (12). The transition regions between subdomains in which either of the two alternatives holds would have to be studied by boundary-layer-type matched asymptotic expansions. This is beyond the scope of the present paper.

4.2.2 Moisture transport in nearly saturated air

In this regime we know that $\delta q_{\mathrm{v}} \equiv 0$, i.e., that $q_{\mathrm{v}}^{(0)} \equiv q_{\mathrm{vs}}^{(0)}$. From the asymptotics of the mixing ratio in (60) we know that $q_{\mathrm{vs}}^{(0)}$ is a function of z only. Therefore, we find from the water vapor transport equation at order $O(\varepsilon)$ that

$$\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q_{\mathrm{v}}^{(0)} \equiv 0\,. \tag{87}$$

Anticipating that turbulent transport will not become important at this order of the perturbation analysis, we conclude that condensation–evaporation of cloud water occurs at an even higher order,

$$C_{\mathrm{d}}^{(0)} = 0 \quad \text{or} \quad \delta q_{\mathrm{v}}^{(n)} \equiv 0\,. \tag{88}$$

In the sequel we let

$$n^* = n + 1 \tag{89}$$

to abbreviate the notation.

Furthermore, from $(45)_2$ at order $O(\varepsilon^{-1})$, and knowing from the above estimates that $\varepsilon^{-n} C_{\mathrm{d}} = O(1)$, we conclude that

$$C_{\mathrm{cr}}^{(-1)} = C_{\mathrm{cr}}^{**} q_{\mathrm{c}}^{(0)} q_{\mathrm{r}}^{(0)} \equiv 0\,. \tag{90}$$

This implies that either $q_{\mathrm{c}}^{(0)} \equiv 0$ or $q_{\mathrm{r}}^{(0)} \equiv 0$. We focus here on the former case, which is valid for precipitating clouds, and let

$$q_{\mathrm{c}}^{(0)} \equiv 0 \tag{91}$$

from here on, so that the cloud water content is systematically small. The second alternative needs to be studied in more detail for non-precipitating clouds and for the upper cloud top, where the total amount of precipitation, q_{r}, is necessarily small. The construction of a related multi-layer model will be addressed in future work.

After these preliminaries we find a set of simplified asymptotic equations for $\delta q_{\mathrm{v}}^{(n^*)}$, $q_{\mathrm{c}}^{(1)}$, $q_{\mathrm{r}}^{(0)}$,

$$\begin{aligned} w^{(0)} \frac{\mathrm{d} q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} &= -C_{\mathrm{d}}^{(0)} + D_{q_{\mathrm{v}}}^{(0)} \\ \left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q_{\mathrm{c}}^{(1)} &= C_{\mathrm{d}}^{(0)} - C_{\mathrm{cr}}^{(0)} \\ \left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q_{\mathrm{r}}^{(0)} &= 0 \end{aligned} \tag{92}$$

where

$$\begin{aligned} C_{\mathrm{d}}^{(0)} &= C_{\mathrm{d}}^{**}\; \delta q_{\mathrm{v}}^{(n^*)} (q_{\mathrm{c}}^{(1)} + q_{cn}^{**})\; H_{\geq}(q_{\mathrm{c}}^{(1)})\,, \\ C_{\mathrm{cr}}^{(0)} &= C_{\mathrm{cr}}^{**} q_{\mathrm{c}}^{(1)} q_{\mathrm{r}}^{(0)}\,. \end{aligned} \tag{93}$$

In the potential temperature equation (80) we need to evaluate the leading-order rain evaporation rate, $C_{\mathrm{ev}}^{(0)}$. However, according to (93) the water vapor mixing ratio deviates from its saturation value only at order ε^{n^*}, so that $C_{\mathrm{ev}}^{(i)} = 0$ for $i \in \{0, \ldots, n^* - 1\}$.

Whereas we found evolution equations for $q_{\mathrm{c}}^{(1)}$ and $q_{\mathrm{r}}^{(0)}$, the water vapor transport equations provided an *implicit definition* of the condensation rate,

$$C_{\mathrm{d}}^{(0)} = -\left[w^{(0)} \frac{\mathrm{d} q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} - D_{q_{\mathrm{v}}}^{(0)} \right]. \tag{94}$$

Through (93) this determines, in turn, the local small deviation from exact saturation, $\delta q_{\mathrm{v}}^{(n^*)}$. We notice in passing, that $C_{\mathrm{d}}^{(0)}$ may have a positive or negative sign, indicating condensation or evaporation of cloud water, respectively. Of course, a negative sign makes sense only as long as long as $q_c^{(1)} > 0$, and this is accounted for by the Heaviside switch, $H_{\geq}(q_c^{(1)})$.

Notice also that $C_d^{(0)}$ is not only determined by vertical motion and the associated change of the saturation water content, but also by (horizontal) turbulent diffusion of water vapor. When this mechanism, near the edge of a region of saturated air, removes water vapor by redistributing it into the neighboring unsaturated region, then the air will become locally undersaturated, and any available cloud water will evaporate. As noted in [8], this mechanism can also be invoked by *numerical* diffusion in cloud resolving models, leading to artificial effects near cloud boundaries.

4.2.3 Undersaturated air

In undersaturated air $q_c = 0$, and the condensation–evaporation source term for the vapor-to-cloud water transition, $C_d^{(0)}$, vanishes. In this regime we do have a nonzero evaporation of precipitation, $C_{ev}^{(0)} = C_{ev}^{**}\left(q_{vs}^{(0)}(z) - q_v^{(0)}\right) q_r^{(0)\,\frac{1}{2}} \neq 0$, and a nontrivial source term for potential temperature remains in (80). However, with the present scalings evaporation is too weak to affect precipitation and water vapor content at leading order, and we obtain, besides $q_c^{(1)} \equiv 0$, the two homogeneous transport equations,

$$\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q_v^{(0)} = \left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q_r^{(0)} = 0\,. \tag{95}$$

This completes the derivation of the simplified asymptotic equations for the present asymptotic regime except for the separation of the convective-scale and bulk microscale contributions of the various equations, and a discussion of the turbulent transport terms.

Separation of long-wave and short-wave components is achieved in a standard fashion by imposing sublinear growth conditions. Consider, e.g., the horizontal momentum equation in (76). Since $\boldsymbol{u}^{(0)}$, $p^{(3)}$ do not depend on $\boldsymbol{\eta}$, and since we may assume $\boldsymbol{D}_{\boldsymbol{u}}^{(-1)}$ to be a horizontal divergence, e.g., $\boldsymbol{D}_{\boldsymbol{u}}^{(-1)} = \nabla_{\boldsymbol{\eta}} \cdot \left(K_t \nabla_{\boldsymbol{\eta}} \boldsymbol{u}\right)$, integration in $\boldsymbol{\eta}$ over a large domain Ω yields

$$\rho_0\, \boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\, p^{(3)} = -\frac{1}{|\Omega|} \oint\limits_{\partial\Omega} \left(p^{(4)}\mathbf{1} - K_t \nabla_{\boldsymbol{\eta}} \boldsymbol{u}\right) \cdot \boldsymbol{n}\, \mathrm{d}\sigma\,, \tag{96}$$

where $\mathbf{1}$ is the two-dimensional unit tensor. Using the standard argument of sublinear growth of the perturbation functions in the integrand with respect to $|\boldsymbol{\eta}|$ we conclude that the integral cannot grow as fast as $|\Omega|$ for increasing size of the domain (vanishing surface-to-volume ratio). In the limit we find $\rho_0\, \boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\, p^{(3)} = 0$, which is equivalent to the mesoscale horizontal momentum equation in $(71)_1$. For the potential temperature equation we proceed analogously.

A detailed discussion of how to incorporate the turbulent transport terms in the present asymptotic framework is beyond the scope of the present paper. To obtain an idea of how one may want to proceed the reader may consult Ref. [15]. In this paper, Ekman boundary-layer theory, which prominently includes the effects of turbulence, is reconsidered via multiple-scale asymptotics.

5 Gravity-wave generation by moist convective processes

5.1 Scalings, asymptotic ansatz, and key results

In the present section we summarize the main results of the asymptotic analysis to provide a compact overview. Details of the derivations are given in subsequent sections.

Here we describe how 10 km convective-scale anelastic flows may interact with 70, ..., 100 km mesoscale gravity waves on time scales of about 20 min. The latter corresponds to the time scale of advection on the smaller scale, given characteristic flow velocities of 10 m/s. At the same time, it is the characteristic time scale for longer-wavelength internal gravity waves, given a typical buoyancy frequency. Thus we consider a single time, multiple-space-scale asymptotic ansatz,

$$\mathbf{U}(\boldsymbol{x}, z, t; \varepsilon) = \sum_i \varepsilon^i\, \mathbf{U}^{(i)}(\boldsymbol{x}, \boldsymbol{\xi}, z, t)\,, \tag{97}$$

where

$$\boldsymbol{\xi} = \varepsilon \boldsymbol{x} \tag{98}$$

is the horizontal coordinate that resolves the mesoscales.

The large-scale dynamics is governed by a set of linearized internal wave equations with nontrivial driving terms for momentum and potential temperature.

Mesoscale wave dynamics with upscale momentum transport

$$\overline{\boldsymbol{u}}_t + \nabla_{\boldsymbol{\xi}}\, \pi^{(3)} = -\partial_z\left(\overline{w\boldsymbol{u}}\right), \quad \Theta_t^{(3)} + \overline{w^{(1)}}\frac{d\Theta_2}{dz} = \overline{S_\theta^{(3)}}, \quad \partial_z \pi^{(3)} = \Theta^{(3)}, \quad \rho_0 \nabla_{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{u}} + \partial_z\left(\rho_0 \overline{w^{(1)}}\right) = 0\,. \tag{99}$$

Here, with the decomposition into latent heat and external diabatic effects, the third-order potential temperature source term reads

$$S_\theta^{(3)} = \tilde{S}_\theta^{(3)} + \frac{\Gamma^{**} L^{**} q_{\text{vs}}^{**}}{p_0}\left(C_{\text{d}}^{(1)} - C_{\text{ev}}^{(1)}\right). \tag{100}$$

In addition, the leading-order vertical velocity must satisfy the

Constraint of organized convection

$$(\overline{w})^{(0)} = 0\,. \tag{101}$$

This constraint can be satisfied in two ways. Either, $w \neq 0$ only within small horizontal subdomains so that the vertical mass flux merely accumulates to a higher-order mean flux, or – in the case of what one may call organized convection – the regions with leading-order vertical motions are densely packed, but the vertical motions from up and downdrafts cancel on average. We keep track of these two different regimes, which differ in terms of their moisture content in under-saturated air, and discuss them in the context of Eqs. (104) and (105) below.

We notice that the Eqs. (99) allow for steady vertically sheared horizontal flows with $\overline{\boldsymbol{u}} = \overline{\boldsymbol{u}}(\boldsymbol{\xi}, z)$, $\nabla_{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{u}} \equiv 0$, and $\overline{w^{(1)}} \equiv 0$, when both driving terms, $(\overline{wu})_z$ and $\overline{S_\theta^{(3)}}$ in the first two equations vanish. Thus, besides unsteady internal waves, the range of solutions of these equations also includes large-scale organized quasi-steady flow patterns whose structure is governed by nonzero but quasi-steady source terms (see also [19–21]).

The mesoscale waves interact with convective-scale anelastic dynamics involving a fully nonlinear horizontal momentum balance,

Convective-scale horizontal dynamics

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + w \boldsymbol{u}_z + \nabla_{\boldsymbol{x}} \pi^{(4)} &= -\nabla_{\boldsymbol{\xi}} \pi^{(3)} + D_u\,. \\ \nabla_{\boldsymbol{x}} \cdot (\rho_0 \boldsymbol{u}) + (\rho_0 w)_z &= 0\,, \end{aligned} \tag{102}$$

and a nonlinear control of the vertical motions by moist processes. In the context of (64) we have argued that typical atmospheric stratifications are neutrally stable, at least up to second order in ϵ. Working from the general assumption that the third-order θ distribution – accounting for latent-heat release – *is* stable, i.e.,

$$\theta_z^{(3)} + \frac{\Gamma^{**} L^{**} q_{\text{vs}}^{**}}{p_0}\frac{dq_{\text{vs}}^{(1)}}{dz} \geq 0\,, \tag{103}$$

we have the following results.

Moisture transport and nonlinear control of vertical motions

Strongly undersaturated air: ($q_{\mathrm{vs}}^{(0)} - q_{\mathrm{v}}^{(0)} > 0$)

$$
\begin{aligned}
q_{\mathrm{v}\ t}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(0)} + w^{(0)} q_{\mathrm{v}\ z}^{(0)} &= C_{\mathrm{ev}}^{(0)} \\
q_{\mathrm{r}\ t}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{r}}^{(0)} + w^{(0)} q_{\mathrm{r}\ z}^{(0)} + \frac{V_{\mathrm{T}}^{**}}{\rho_0} \left(\rho_0 q_{\mathrm{r}}^{(0)}\right)_z &= -C_{\mathrm{ev}}^{(0)} \\
w^{(0)} &= -\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{(p_0 d\Theta_2/dz)(z)} C_{\mathrm{ev}}^{(0)} \\
C_{\mathrm{ev}}^{(0)} &= C_{\mathrm{ev}}^{**} \left(q_{\mathrm{vs}}^{(0)}(z) - q_{\mathrm{v}}^{(0)}\right) q_{\mathrm{r}}^{(0)\,\frac{1}{2}}
\end{aligned}
\tag{104}
$$

This is the interesting regime, in which strong organized convection is possible. Here, we have $\overline{w^{(0)}} \equiv 0$, but may have nonzero leading-order vertical velocities everywhere with downdrafts from evaporative cooling balancing updrafts of saturated, condensing, and strongly precipitating air. These strong vertical flows will induce the convection-related effective source term in the mesoscale horizontal momentum equation, and this is one of the interesting and important effects revealed here.

Unfortunately, the present multiple-scale theory is not closed for this regime: to determine the effective averaged source term for the third-order mesoscale potential temperature distribution in (99) we need explicit expressions for the first-order condensation and evaporation source terms $C_{\mathrm{d}}^{(1)}$ and $C_{\mathrm{ev}}^{(1)}$ according to (100). The condensation term is readily available from (108) below.

However, the first-order *evaporation* term involves a term $C_{\mathrm{ev}}^{**}\left(q_{\mathrm{vs}}^{(1)}(z) - q_{\mathrm{v}}^{(1)}\right) q_{\mathrm{r}}^{(0)\,\frac{1}{2}}$, and its determination would require the solution of the equation for $q_{\mathrm{v}}^{(1)}$ in the strongly undersaturated air region. It is easily seen from $(104)_2$ that the next-order equation would involve the first-order perturbation velocity $\boldsymbol{u}^{(1)}$, etc., rendering the hierarchy of equations unclosed.

We leave a resolution of this issue for future work and consider here the somewhat simpler regime of weakly undersaturated air.

Weakly undersaturated air: ($q_{\mathrm{vs}}^{(0)} - q_{\mathrm{v}}^{(0)} = 0$, $\tilde{S}_\theta^{\,(3)} - \theta_t^{(3)} < 0$)

$$
\begin{aligned}
q_{\mathrm{v}\ t}^{(1)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(1)} + w^{(1)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} &= C_{\mathrm{ev}}^{(1)} \\
q_{\mathrm{r}\ t}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{r}}^{(0)} + \frac{V_{\mathrm{T}}^{**}}{\rho_0} \left(\rho_0 q_{\mathrm{r}}^{(0)}\right)_z &= 0 \\
w^{(1)} &= \left(-\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{(p_0)(z)} C_{\mathrm{ev}}^{(1)} + \left(\tilde{S}_\theta^{\,(3)} - \theta_t^{(3)}\right)\right) \left(\frac{\mathrm{d}\Theta_2}{\mathrm{d}z}\right)^{-1} \\
C_{\mathrm{ev}}^{(1)} &= C_{\mathrm{ev}}^{**} \left(q_{\mathrm{vs}}^{(1)}(z) - q_{\mathrm{v}}^{(1)}\right) q_{\mathrm{r}}^{(0)\,\frac{1}{2}}
\end{aligned}
\tag{105}
$$

This regime is intriguing in a different way than that for strongly undersaturated air. Here, leading-order vertical downdrafts are suppressed, whereas in saturated air, leading-order vertical motions may occur but must be directed upwards. As a consequence, the sublinear growth constraint $\overline{w^{(0)}} \equiv 0$ from (101) can only be satisfied if *either* there are no vertical leading-order motions at all (this would be the conclusion in classical multiple-scale analyses), *or* if the occurrence of strong precipitating updrafts is restricted to a total horizontal area of relative size of order $O(\varepsilon)$ as $\varepsilon \to 0$. Such a regime is nonclassical in the context of multiple-scale asymptotics and it will require the combination of multiscale asymptotics with concepts from stochastic modeling to obtain a meaningful closed model. In the latter case, the first-order mean vertical velocity, $\overline{w^{(1)}}$, will be composed of two nontrivial contributions, namely the classical average of the first-order perturbation velocity, and the resulting first-order mean from sparsely distributed columns with leading-order updrafts.

Saturated air:($q_{\mathrm{vs}}^{(0)} - q_{\mathrm{v}}^{(0)} = 0$)

$$
q_{\mathrm{r}\ t}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{r}}^{(0)} + w^{(0)} q_{\mathrm{r}\ z}^{(0)} + \frac{1}{\rho_0} \left(\rho_0 V_T q_{\mathrm{r}}^{(0)}\right)_z = -w^{(0)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z}\,. \tag{106}
$$

$$w^{(0)} = \begin{cases} \left(\tilde{S}_\theta^{(3)} - \theta_t^{(3)}\right) \left(\partial\theta^{(3)}/\partial z + \Gamma^{**} L^{**} q_{\mathrm{vs}}^{**} \partial q_{\mathrm{vs}}^{(1)}/\partial z\right)^{-1} & \text{for} \quad \tilde{S}_\theta^{(3)} - \theta_t^{(3)} > 0 \\ 0 & \text{for} \quad \tilde{S}_\theta^{(3)} - \theta_t^{(3)} \le 0 \,, \end{cases} \tag{107}$$

where $\tilde{S}_\theta^{(3)}$ denotes all third-order diabatic effects except for latent-heat conversion, which is captured through the vertical derivative of $q_{\mathrm{vs}}^{(1)}$. Here we observe how external diabatic source terms, $\tilde{S}_\theta^{(3)}$, drive upward motions in weakly moist stable air.

For completeness we note that the relevant contribution from the first-order condensation term $C_{\mathrm{d}}^{(1)}$ in (100) reads

$$\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0} w^{(0)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(1)}}{\mathrm{d}z} \,. \tag{108}$$

In contrast, the term$(\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}/p_0)\, w^{(1)} \mathrm{d}q_{\mathrm{vs}}^{(0)}/\mathrm{d}z$ cancels with the third-order vertical advection term for potential temperature, $w^{(1)}\mathrm{d}\Theta_2/\mathrm{d}z$ in the evolution equation for $\theta^{(3)}$.

5.2 Key steps of the derivations

Here are the leading few raw equations resulting from insertion of the expansion scheme in (97) into the governing equations from Sect. 3.3, collecting like powers of ε, and anticipating that the leading-order pressure and density are time independent.

Mass:

$$\begin{aligned} \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(0)} + (\rho w)_z^{(0)} &= 0 \\ \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(1)} + (\rho w)_z^{(1)} &= -\nabla_{\boldsymbol{\xi}} \cdot (\rho \boldsymbol{u})^{(0)} \end{aligned} \tag{109}$$

Horizontal momentum:

$$\begin{aligned} \nabla_{\boldsymbol{x}} p^{(0)} &= 0 \\ \nabla_{\boldsymbol{x}} p^{(j+1)} + \nabla_{\boldsymbol{\xi}} p^{(j)} &= 0 \qquad (j \in \{0, 1, 2\}) \\ (\rho \boldsymbol{u})_t^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} \circ \boldsymbol{u})^{(0)} + (\rho w \boldsymbol{u})_z^{(0)} + \nabla_{\boldsymbol{x}} p^{(4)} + \nabla_{\boldsymbol{\xi}} p^{(3)} &= \boldsymbol{D}_{\rho \boldsymbol{u}}^{(0)} \,. \end{aligned} \tag{110}$$

Vertical momentum:

$$\partial_z p^{(j)} = -\rho^{(j)} \quad (j \in \{0, 1, 2, 3\}) \tag{111}$$

Potential temperature:

$$\begin{aligned} S_\theta^{(0)} &= 0 \,, \\ S_\theta^{(1)} &= 0 \,, \\ w^{(0)} \frac{d\Theta_2}{dz} &= S_\theta^{(2)} \,, \\ \theta_t^{(3)} + w^{(0)} \theta_z^{(3)} + w^{(1)} \frac{d\Theta_2}{dz} &= S_\theta^{(3)} \,, \end{aligned} \tag{112}$$

In these equations $(ab)^{(1)} = a^{(1)} b^{(0)} + a^{(0)} b^{(1)}$. In deriving the equations in (112) we have anticipated the expansion scheme for potential temperature

$$\theta = 1 + \varepsilon^2 \Theta_2(z) + \varepsilon^3 \theta^{(3)}(\boldsymbol{\xi}, z, t) + \varepsilon^3 \theta^{(4)}(\boldsymbol{x}, \boldsymbol{\xi}, z, t) \ldots . \tag{113}$$

Its validity may be verified by combining the leading vertical and horizontal momentum balances and the expansion of the equation of state, $p = (\rho\theta)^\gamma$, with sublinear growth conditions for $p^{(i)}$ for $(i \in \{0, 1, 2\})$ in terms of the short-scale horizontal coordinate, $\boldsymbol{x}$. In this process, the expansion of the equation of state will be analogous to that of the temperature definition in (50) and (51).

Water vapor:

$$
\begin{aligned}
C_{\mathrm{d}}^{(i)} &= 0 \qquad \text{for} \quad (i \in \{-n, \ldots, -1\})\,, \\
q_{\mathrm{v}\,t}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(0)} + w^{(0)} q_{\mathrm{v}\,z}^{(0)} &= -C_{\mathrm{d}}^{(0)} + C_{\mathrm{ev}}^{(0)}\,. \\
q_{\mathrm{v}\,t}^{(1)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(1)} + \boldsymbol{u}^{(1)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(0)} + w^{(0)} q_{\mathrm{v}\,z}^{(1)} + w^{(1)} q_{\mathrm{v}\,z}^{(0)} &= -C_{\mathrm{d}}^{(1)} + C_{\mathrm{ev}}^{(1)}\,.
\end{aligned}
\tag{114}
$$

Cloud water:

$$
\begin{aligned}
C_{\mathrm{cr}}^{(-1)} &= 0\,, \\
q_{\mathrm{c}\,t}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{c}}^{(0)} + w^{(0)} q_{\mathrm{c}\,z}^{(0)} &= C_{\mathrm{d}}^{(0)} - C_{\mathrm{cr}}^{(0)}\,.
\end{aligned}
\tag{115}
$$

Rain water:

$$
q_{\mathrm{r}\,t}^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{r}}^{(0)} + w^{(0)} q_{\mathrm{r}\,z}^{(0)} + \frac{1}{\rho_0}\left(\rho_0 V_T q_{\mathrm{r}}^{(0)}\right)_z = C_{\mathrm{cr}}^{(0)} - C_{\mathrm{ev}}^{(0)}\,. \tag{116}
$$

5.2.1 Nonlinear control of vertical motions

Equation $(112)_3$ states that, in a stably stratified atmosphere, the vertical velocity is directly proportional to the rate of diabatic heating. However, in the present context such a statement captures only part of the essence of the prevailing balances:

The strongest source term for potential temperature is caused by latent-heat conversion. Its strength is of order $O(\varepsilon^2)$, so that it induces the source term $S_\theta^{(2)}$ in $(112)_3$. This order-of-magnitude estimate may be verified by combining (40) with estimates of the characteristic time scales of latent-heat conversion: condensation in vertical updrafts occurs on convective time scales, i.e., on time scales of order $O(1)$ nondimensionally. The nondimensional rate of evaporation of precipitation, C_{ev} in $(45)_{1,3}$, was also assessed to be of order $O(1)$ as $\varepsilon \to 0$.

All other source terms for θ that we have analyzed, such as the effects of radiation and turbulent transport (details not shown due to a lack of space), are one or two orders of magnitude smaller, and we subsume these into the third-order term $S_\theta^{(3)}$ in $(112)_4$. We conclude that the source term $S_\theta^{(2)}$ in $(112)_3$ involves only the effects of latent-heat conversion due to condensation and evaporation of precipitation, so that

$$
\begin{aligned}
S_\theta^{(2)} &= \frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0(z)} \left(-H_{\geq}(q_{\mathrm{v}} - q_{\mathrm{vs}}) w^{(0)} \frac{\mathrm{d} q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} + H_{>}(q_{\mathrm{vs}} - q_{\mathrm{v}})\, C_{\mathrm{ev}}^{(0)} \right), \\
S_\theta^{(3)} &= \frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0(z)} \left(-H_{\geq}(q_{\mathrm{v}} - q_{\mathrm{vs}}) \left(w^{(0)} \frac{\mathrm{d} q_{\mathrm{vs}}^{(1)}}{\mathrm{d}z} + w^{(1)} \frac{\mathrm{d} q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} \right) + H_{>}(q_{\mathrm{vs}} - q_{\mathrm{v}})\, C_{\mathrm{ev}}^{(1)} \right) + \tilde{S}_\theta^{\,(3)}\,.
\end{aligned}
\tag{117}
$$

The terms involving $H_{\geq}$ become relevant in saturated air, and they are a consequence of (114). In saturated air, we have $q_{\mathrm{v}} = q_{\mathrm{vs}}$ up to very high order in ε, see (74). From (60), $q_{\mathrm{vs}}^{(0)}$ is a function of z only, and this is true also for the next-order term, $q_{\mathrm{vs}}^{(1)}(z)$. As a consequence, in deriving (117) from (114), the horizontal advection terms dropped out.

We analyze the consequences of these formulations when inserted in $(112)_{3,4}$ and have to distinguish the three regimes of: (i) saturated air with $q_{\mathrm{v}} \geq q_{\mathrm{vs}}$, (ii) strongly undersaturated air with $q_{\mathrm{vs}}^{(0)}(z) > q_{\mathrm{v}}^{(0)}$, and (iii) weakly undersaturated air with $q_{\mathrm{v}}^{(0)} = q_{\mathrm{vs}}^{(0)}(z)$ but $q_{\mathrm{vs}}^{(1)}(z) > q_{\mathrm{v}}^{(1)}$.

Strongly undersaturated Air In undersaturated air, with $H_{\geq}(\cdot) = 0$ and $H_{>}(\cdot) = 1$, and with second-order deviations from saturation, i.e., $q_{\mathrm{vs}}^{(0)} - q_{\mathrm{v}}^{(0)} > 0$, equation $(112)_3$ directly determines the vertical velocity in the sense of the weak temperature gradient (WTG) approximation

$$
w^{(0)} = -\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**} C_{\mathrm{ev}}^{**}}{(p_0 \mathrm{d}\Theta_2/\mathrm{d}z)(z)} \left(q_{\mathrm{vs}}^{(0)}(z) - q_{\mathrm{v}}^{(0)} \right) q_{\mathrm{r}}^{(0)\,\frac{1}{2}} \quad \text{for } q_{\mathrm{v}}^{(0)} < q_{\mathrm{vs}}^{(0)}(z)\,. \tag{118}
$$

WTG approximations assume weak horizontal gradients of (potential) temperature on meso- and larger scales, these being enforced by the dominance of gravity and rapid equilibration due to internal gravity waves. If horizontal gradients of temperature are bound to be small, however, then local heating, e.g., due to latent-heat release or radiation with tend to disturb this balance. It can then be maintained only by moving the affected air parcel vertically towards the very level where its temperature again matches that of its environment. Thus, the vertical velocity of a parcel of air is determined by its diabatic heating rate.

In particular, we have an explicit description of downdrafts induced by evaporative cooling, and we conclude that leading-order vertical motions are suppressed entirely in the absence of precipitation.

Weakly undersaturated air In the regime of weakly undersaturated air, in which $q_{\mathrm{v}}^{(0)} = q_{\mathrm{vs}}^{(0)}(z)$ but $q_{\mathrm{vs}}^{(1)}(z) - q_{\mathrm{v}}^{(1)} > 0$, we have, from $(112)_{3,4}$,

$$
\begin{aligned}
w^{(0)} &= 0 \\
w^{(1)} &= \left(-\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**} C_{\mathrm{ev}}^{**}}{p_0} \left(q_{\mathrm{vs}}^{(1)}(z) - q_{\mathrm{v}}^{(1)} \right) q_{\mathrm{r}}^{(0)\,\frac{1}{2}} + \left(\tilde{S}_{\theta}^{\,(3)} - \theta_t^{(3)} \right) \right) \left(\frac{\mathrm{d}\Theta_2}{\mathrm{d}z} \right)^{-1} .
\end{aligned}
\tag{119}
$$

Here $\tilde{S}_{\theta}^{\,(3)}$ collects all third-order potential temperature source terms except for the effects of latent-heat conversion.

Saturated Air In saturated air, with $H_{\geq}(\cdot) = 1$ and $H_{>}(\cdot) = 0$ in (117), the situation is entirely different. Here, $(112)_3$ in combination with (94) for the condensation rate yields

$$
w^{(0)} \frac{\mathrm{d}\Theta_2}{\mathrm{d}z} = -w^{(0)} \frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0(z)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} . \tag{120}
$$

Notice that we have dropped the turbulent transport term from (94). Estimates in [15] show that turbulent transport becomes effective at much smaller scales only. Clearly (120) can be satisfied if we either have a moist adiabatic stratification so that

$$
\frac{\mathrm{d}\Theta_2}{\mathrm{d}z} = -\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0} \frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} \quad \text{or } w^{(0)} \equiv 0 . \tag{121}
$$

As argued in Sect. 3.4.3 we do assume moist adiabatic stratification to the order in ε considered here, so that the former alternative holds. As a consequence, $w^{(0)}$ cancels from (120), and in saturated air the WTG-type equation in $(112)_3$ does not determine the leading-order vertical velocity. Instead it is determined by higher-order potential temperature equations. We will discuss situations with $O(\varepsilon^3)$-deviations from moist adiabatic stratification below. In this case, the third-order equation in $(112)_4$ determines the vertical motion as we will see shortly.

Whatever the results of the related derivations, they must observe additional constraints induced by the singular structure of the cloud water and precipitation evolution equations. First we conclude from the leading-order cloud water equation in $(115)_1$ and from the definition of $\hat{C}_{\mathrm{cr}}$ in $(45)_3$ that either $q_{\mathrm{c}}^{(0)} = 0$ or $q_{\mathrm{r}}^{(0)} = 0$. As we can see in (116), the second option would induce $C_{\mathrm{cr}}^{(0)} = 0$ so that $q_{\mathrm{r}}^{(1)} = 0$, too. This regime corresponds to non-precipitating clouds (absence of rain), and will not be considered further here. The alternative, $q_{\mathrm{c}}^{(0)} = 0$, must hold in precipitating clouds. In this regime, $(115)_2$ yields a balance between condensation and the formation of precipitation, so that

$$
C_{\mathrm{d}}^{(0)} - C_{\mathrm{cr}}^{(0)} = -w^{(0)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} - C_{\mathrm{cr}}^{**}\, q_{\mathrm{c}}^{(1)} q_{\mathrm{r}}^{(0)\,1/2} = 0 \quad \text{or } w^{(0)} = -\frac{C_{\mathrm{cr}}^{**}\, q_{\mathrm{c}}^{(1)} q_{\mathrm{r}}^{(0)\,1/2}}{\mathrm{d}q_{\mathrm{vs}}^{(0)}/\mathrm{d}z} . \tag{122}
$$

For given $w^{(0)} \geq 0$ and $q_{\mathrm{r}}^{(0)}$ this determines the local amount of cloud water, $q_{\mathrm{c}}^{(1)}$. However, as $\mathrm{d}q_{\mathrm{vs}}^{(0)}/\mathrm{d}z < 0$ and $q_{\mathrm{c}}^{(1)}, q_{\mathrm{r}}^{(0)} \geq 0$ we also have a nonlinear constraint on the vertical velocity,

$$
w^{(0)} \geq 0 , \tag{123}
$$

which is valid in saturated air in the presence of precipitation.

To make progress in determining the leading- and first-order vertical velocity in saturated air, we consider the third-order potential temperature equation in $(112)_4$. We divide the source term into contributions from latent-heat conversion and another term that collects all other effects at that order. Leaving aside the rain evaporation effect, because we consider saturated air, we have

$$S_\theta^{(3)} = \frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0} C_{\mathrm{d}}^{(1)} + \tilde{S}_\theta^{(3)} . \tag{124}$$

From the third-order water vapor balance, and using the fact that deviations from saturation occur at very high order only, so that $q_{\mathrm{v}}^{(1)} = q_{\mathrm{vs}}^{(1)}(z)$, we conclude from $(114)_3$ that

$$C_{\mathrm{d}}^{(1)} = -w^{(1)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} - w^{(0)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(1)}}{\mathrm{d}z} . \tag{125}$$

Then the third-order potential temperature balance, $(112)_4$, reads

$$\theta_t^{(3)} + w^{(0)} \theta_z^{(3)} + w^{(1)} \frac{\mathrm{d}\Theta_2}{\mathrm{d}z} = -\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0} \left(w^{(1)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(0)}}{\mathrm{d}z} + w^{(0)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(1)}}{\mathrm{d}z} \right) + \tilde{S}_\theta^{(3)} . \tag{126}$$

As we consider here the regime of moist adiabatic stratification at order $O(\varepsilon^2)$ from $(121)_1$ we find

$$\theta_t^{(3)} + w^{(0)} \theta_z^{(3)} = -\frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0} w^{(0)} \frac{\mathrm{d}q_{\mathrm{vs}}^{(1)}}{\mathrm{d}z} + \tilde{S}_\theta^{(3)} . \tag{127}$$

Since $\theta^{(3)}$ does not depend on the convective-scale variable $\boldsymbol{x}$ [see (113)], it is clear that its time derivative, $\theta_t^{(3)}$, will have to be determined from sublinear growth conditions below. For given $\theta_t^{(3)}$ and a moist stable stratification at the given order we obtain the determining equation for the leading-order vertical velocity,

$$w^{(0)} = \left(\tilde{S}_\theta^{(3)} - \theta_t^{(3)} \right) \left(\theta_z^{(3)} + \frac{\Gamma^{**} L^{**} q_{\mathrm{vs}}^{**}}{p_0} \frac{\mathrm{d}q_{\mathrm{vs}}^{(1)}}{\mathrm{d}z} \right)^{-1} . \tag{128}$$

There is a catch, however, as we have previously found the constraint of positivity for $w^{(0)}$ in (123). Thus, this last equation holds only in saturated air when the right-hand side is positive. Should the right-hand side become negative locally, the ensuing downward motion would immediately (relative to the considered time scale) lead to evaporation of the local first-order cloud water content, leaving us with weakly undersaturated air and first-order vertical velocities determined by (119).

5.2.2 Sublinear growth conditions

In (97) we have set up an expansion scheme that involves multiple horizontal scales. To determine the dependence of the expansion functions $\mathbf{U}^{(i)}(\boldsymbol{x}, \boldsymbol{\xi}, z, t)$ on the large-scale spacial coordinate, $\boldsymbol{\xi}$, we employ sublinear growth conditions (see e.g., [17] for an explanation in the context of meteorological modeling).

Mesoscale mass balance and the notion of organized convection Averaging $(109)_{1,2}$ in the fast horizontal coordinate $\boldsymbol{x}$ we find

$$\begin{aligned} (\rho_0 \overline{w^{(0)}})_z &= 0 \\ \rho_0 \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{u}^{(0)} + (\rho_0 \overline{w^{(1)}})_z &= 0 . \end{aligned} \tag{129}$$

With the large-scale bottom boundary condition $\overline{w^{(0)}}\Big|_{z=0} = 0$, the first equation immediately yields

$$\overline{w^{(0)}} = 0 . \tag{130}$$

This is the organized convection constraint from (101)

Equation (129) is a standard anelastic constraint for the large-scale averaged motion.

Mesoscale momentum balance and convection-induced mean forcing Averaging $(110)_{1,2}$ in $\boldsymbol{x}$ we find

$$\nabla_{\boldsymbol{x}} p^{(i)} = 0 \quad \text{for} \quad (i \in \{0,1,2,3\}) \quad \text{and} \quad \nabla_{\boldsymbol{\xi}} p^{(i)} = 0 \quad \text{for} \quad (i \in \{0,1,2\})\,. \tag{131}$$

The mesoscale evolution of the velocity field is obtained analogously from $(110)_3$,

$$\overline{\boldsymbol{u}^{(0)}}_t + (\overline{w\boldsymbol{u}})^{(0)}_z + \nabla_{\boldsymbol{\xi}} p^{(3)} = 0\,. \tag{132}$$

Here we have assumed that the transport term $\boldsymbol{D}_{\rho \boldsymbol{u}}$ has the form of a small-scale divergence, so that it cancels in the horizontal average due to a surface-to-volume-ratio argument.

Notice, in particular, the appearance of the term $(\overline{w\boldsymbol{u}})^{(0)}_z$ which indicates the net vertical transport of horizontal momentum *at leading order*. In standard scale analysis for the mesoscales one would have assumed that $|w|/|\boldsymbol{u}| \sim h/L = O(\varepsilon)$, where h, L denote the characteristic vertical and horizontal scales, respectively, and where we have used the present asymptotic scaling $\boldsymbol{\xi} = \varepsilon \boldsymbol{x}$ for the estimate in terms of ε. As a consequence, in such an analysis $w^{(0)} \equiv 0$, and $w = \varepsilon w^{(1)} + o(\varepsilon)$. The mentioned transport term would be absent in this case.

In contrast, following the discussion of the mesoscale mass balance, we have here two multiscale regimes in which leading-order vertical velocities can develop on the small scale. In the first regime with sparse distribution of vertical convection sites we have $\overline{w\boldsymbol{u}} = O(\varepsilon)$. However, in the regime of organized convection $\overline{w\boldsymbol{u}} = O(1)$ and there will be a nontrivial contribution to the horizontal momentum balance.

From Sect. 5.2.1 it is clear that vertical motions are controlled by an interplay of moisture transport and other diabatic effects, so that there is a strong coupling between mesoscale horizontal motions and the small-scale convective activity.

Mesoscale vertical momentum balance – hydrostatics Averaging (111) in $\boldsymbol{x}$ does not yield new information. All pressure variables, $p^{(i)}$ for $i \in \{0,1,2,3\}$ are in hydrostatic balance.

Mesoscale potential temperature transport, organized convection, and internal waves Averaging $(112)_3$ and using $\overline{w^{(0)}} = 0$ from (130) we obtain a constraint on the second-order diabatic source term $S_\theta^{(2)}$,

$$\overline{S_\theta^{(2)}} = \frac{\Gamma^{**} L^{**} q_{\text{vs}}^{**}}{p_0} \left(\overline{H_{\geq}(\delta q_{\text{v}})\, w^{(0)} \frac{\mathrm{d}q_{\text{vs}}^{(0)}}{\mathrm{d}z} + H_{>}(-\delta q_{\text{v}})\, C_{\text{ev}}^{(0)}} \right) = 0\,. \tag{133}$$

Using the determining equation for $w^{(0)}$ for saturated air in (128) we have

$$\overline{H_{\geq}(\delta q_{\text{v}}) \frac{\tilde{S}_\theta^{\,(3)} - \theta_t^{(3)}}{\theta_z^{(3)} + \frac{\Gamma^{**} L^{**} q_{\text{vs}}^{**}}{p_0} \frac{\mathrm{d}q_{\text{vs}}^{(1)}}{\mathrm{d}z}} \frac{\mathrm{d}q_{\text{vs}}^{(0)}}{\mathrm{d}z} + H_{>}(-\delta q_{\text{v}})\, C_{\text{ev}}^{**} (q_{\text{vs}}^{(0)} - q_{\text{v}}^{(0)}) q_{\text{r}}^{(0)\,1/2}} = 0\,. \tag{134}$$

Again, such a constraint can be satisfied if leading-order motions are sparsely distributed and the average merely accumulates a higher-order perturbation. The more interesting alternative is organized convection, in which vertical motions do occur over an order $O(1)$ fraction of the domain, but the driving diabatic sources and sinks are organized spatially such that cancelation of updrafts and downdrafts occurs in the mean.

At the next order the sublinear growth condition for (112) yields the mesoscale potential temperature evolution equation,

$$\theta_t^{(3)} + \overline{w^{(1)}} \frac{\mathrm{d}\Theta_2}{\mathrm{d}z} = \overline{S_\theta^{(3)}}\,. \tag{135}$$

This completes the derivation of the simplified meso-convective multiscale equations from (99)–(107).

6 Conclusions

In this paper we have demonstrated how typical bulk microphysics closure schemes for moist processes can be incorporated systematically in the multi-scale modeling framework for atmospheric flows from [14,17]. The key steps were careful nondimensionalization, and appropriate choices of distinguished limits between various power-law exponents, Damköhler numbers, and activation energy parameters on the one hand, and the unified asymptotic expansion parameter identified in [14] on the other hand.

Two new multi-scale models have been derived in Sects. 4 and 5, which describe the short-time evolution of slender *hot towers* embedded in a convective-scale environment, and organized convection in mesoscale flows, respectively. The first model reveals an interaction of linearized, anelastic, convective-scale motions with bulk microscale columnar flow through nonlinear averages of the moisture source terms. The second model exhibits a similar interaction between mesoscale internal gravity waves and nonlinear, anelastic, moist flow on the convective scales. An important feature of the convective-scale motions is the highly nonlinear control of vertical velocity through interactions of buoyancy effects with latent-heat release. The second class of models has raised some open issues regarding closedness of the obtained equation systems. In the regime of organized convection with leading-order vertical upward and downward motions, higher-order perturbations would be needed to close the model hierarchy. In contrast, in the regime of sparsely distributed updrafts, techniques foreign to classical multiple-scale analyses will have to be invoked in addition to define a closed model for the spatial distribution of the hot towers.

We leave detailed discussions of these issues, (numerical) solutions of the new model equations, and applications in the context of innovative computational modeling strategies for future publications.

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