An explicit example with non-Gaussian probability distribution for nontrivial scalar mean and fluctuation

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Recently, one of the authors, studying a model for turbulent diffusion involving a large-scale velocity field rapidly fluctuating in time, rigorously demonstrated intermittency in a diffusing scalar field by exhibiting broader than Gaussian tails in the scalar PDF. Here, we explore this model further with exact formulas within the context of general initial data possessing both a mean and a fluctuating component. Several new phenomena due to the presence of a nonzero scalar mean are documented here. We will establish that the limiting long time scalar PDF has long tails, as well as persisting skewness. Further, we show that the limiting PDF depends on the large-scale energy of initial temperature fluctuations and exhibits long time memory of the initial data. Additionally, we will exhibit an explicit phase transition occurring in the scalar PDF as this large scale energy is varied, whereby the limiting PDF switches between states arising from deterministic initial data and states dominated by fluctuation. © *1996 American Institute of Physics.* [S1070-6631(96)00502-0]

I. INTRODUCTION

An important current problem in turbulence is the understanding of intermittency. This generally refers to situations where the probability of rare events is large. Experimental examples include intermittency in turbulent vorticity,[1] and, more recently, intermittent effects in thermal convection.[2–4] Experimental results in thermal convection at both moderate and high Reynolds numbers indicate that in situations where the velocity field admits a statistical description with roughly Gaussian fluctuations, the temperature field is seen to be highly non-Gaussian, with a probability distribution function possessing long tails.[2] This experimental observation is responsible for generating a large experimental,[3] computational,[5,6] and theoretical effort,[7–12] attempting to explain this phenomenon. An important goal for this effort is to identify the relevant physical mechanisms necessary to produce this increased probability for rare events. Examples include imposed mean thermal gradients[3,9,10] in the context of a passive scalar, as well as the role of buoyancy for the case of activated scalars.[7,13,14]

In recent work, one of the authors explored these issues within the context of a passive scalar advected by a large-scale shear flow rapidly fluctuating in time.[15] In that work, it was rigorously established that the probability distribution function for the temperature field has broader than Gaussian tails for all positive time, provided that the initial temperature profile is a spatially stratified Gaussian random field. These explicit formulas were obtained through exact statistical moment closure and a connection to N-body quantum mechanics. Further, this broad tailed distribution was observed without subjecting the temperature field to a mean thermal gradient. Given the explicit nature of this theory, such models present the opportunity to develop stringent elementary tests to assess the validity of ideas used in theoretical PDF modeling using mapping closures[11] and other closure approximations.

In this paper, we generalize this model to understand intermittency in the context of a passive scalar possessing both nontrivial mean and fluctuating components. Several new phenomena due to the presence of a nonzero scalar mean are documented here in these models. We will rigorously establish that the presence of both mean and fluctuation in nonstratified initial temperature profiles leads to a new phase transition in the limiting, long time scalar PDF as the large-scale energy in the initial temperature field is varied.

We study the following passive scalar:

$$\frac{\partial T}{\partial t}+v(x,z,t)\,\frac{\partial T}{\partial y}=\bar{\kappa}\,\Delta T, \tag{1}$$

with the following initial data:

$$T|_{t=0}=T_0(x,y).$$

Here, the velocity field is the following shear layer:

$$\mathbf{v}={}^t[0,\gamma(t)x+\bar{\gamma}(t)z], \tag{2}$$

where the random function, $\gamma(t)$, is Gaussian white noise with the following correlation function, with brackets, $\langle\cdot\rangle$, implying averaging over different realizations of the random function, γ,

$$\langle\gamma(t)\gamma(t')\rangle=\delta(t-t'),$$

and the function, $\overline{\gamma}(t)$, is a prescribed deterministic function. The effect of this mean flow, $\overline{\gamma}(t)$, will be omitted from the following discussions until Sec. V. Since the equation in (3) is linear, we remove the constant reference temperature and measure the temperature, T, from zero. Except for the fact that the initial temperature field studied here is general and nonstratified, the model utilized here is the same one used by one of the authors[15] to obtain exact results in earlier work.

We study initial data of the following form:

$$T_0(x,y)=\bar{T}(x,y)+\phi(x,y), \tag{3}$$

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where $\bar{T}(x,y)$ is a deterministic mean profile and $\phi(x,y)$ is a Gaussian random field. We will assume that the deterministic initial data corresponds to some finite-energy perturbation of the constant reference temperature. Consequently, we will assume that the space average of this field, $\int\int \bar{T}(x,y)dx\, dy$, is finite. We comment that the more general initial data, $T|_{t=0}=T_0(x,y,z)$, will result in the same phenomena produced by the data given in (3). For brevity in exposition we take the initial data to be a function of x and y alone.

In subsequent sections, we will demonstrate that long time memory of these initial temperature profiles will set the form of the limiting PDF for temperature fluctuations. For example, we will establish that the asymptotic skewness of the PDF can be nonzero, with a coefficient determined by the initial data. Additionally, at long time, the PDF will be shown to depend upon the correlation scale of initial temperature fluctuations in the random field, ϕ. We take the correlation scale to be a measurement of the amount of energy at large scales in a given quantity and use the term thermal correlation scale to denote the correlation scale of initial temperature fluctuations. We will establish that as the thermal correlation scale is varied, the limiting PDF will dramatically change form between distributions arising from purely fluctuating initial temperature profiles and those arising from purely deterministic initial data. For sufficiently long range correlations, the resulting PDF at long time will be shown to depend only on the fluctuating component of the initial temperature field, ϕ. In the opposite extreme, for sufficiently short range correlations, the limiting PDF will be shown to depend only upon the deterministic initial data, $\bar{T}$. Further, we will verify for the model studied here that the following intuition is correct. Namely, for thermal correlation scales that are comparable with the velocity correlation scale, we expect that at long time the PDF will be Gaussian. However, if the thermal correlation scale is short when measured relative to the velocity correlation scale, the PDF will be highly non-Gaussian. We remark that for the model studied here, the velocity correlation scale is formally infinite. Consequently, verification of this intuition will require exploring initial temperature correlation scales that diverge.

We comment that the new phenomena described above are produced by random advection. The ordinary heat equation with the random initial data given in (3) will not exhibit such a phase transition.

The main results of this paper are presented in Secs. III, IV, and V. In Secs. III and IV, we give explicit formulas for the long time asymptotics for arbitrary moments in the isolated cases when the initial data is either purely deterministic or purely fluctuating. In Sec. III, we will document cases in which the skewness persists for all time. In Sec. IV, we will study the flatness as the thermal correlation scale is varied. Then, in Sec. V we present the limiting long time behavior of the skewness and flatness resulting from the interaction of initial mean and fluctuating temperature profiles. In that section we will demonstrate a rigorous phase transition occurring in these moments as the thermal correlation scale is varied. Remarkably, at the transition value, the limiting PDF for the scalar will be shown to even depend on the Schmidt number.

In Sec. II, we obtain exact moment closure formulas for general initial data following closely the work of one of the authors.[15,16] This closure is obtained through an exact quantum mechanical analogy obtained through manipulation of function space integrals. For the spatially linear shear flow considered here, the associated N-body quantum mechanics problem involves a quadratic potential, and is explicitly solvable.[15,17] Then, with these solution formulas, we will derive explicit expressions for the first four moments of the temperature field. These expressions are rather tedious, but essential, in deriving the new phenomena discussed above. The reader may skip to Sec. III, and refer back to these formulas when necessary. We analyze these explicit formulas in Secs. III, IV, and V, using a combination of asymptotic scaling along with general cluster expansions, to obtain the limiting form of the probability distribution function.

II. REPRESENTATION FORMULAS FOR ARBITRARY MOMENTS

For the discussion below, we omit the effect of the mean shear flow by setting the deterministic function $\overline{\gamma}(t)$ to zero; we will consider its effect in Sec. V.

We take the fluctuating component of the initial data in Eq. (1) to be the following Gaussian random field:

$$\phi(x,y)=\int\int e^{i(\eta x+ky)}E^{1/2}(\eta,k)dW(\eta)\otimes dW(k). \quad (4)$$

Here, the function, $E(k,\eta)$ is an arbitrary spectral energy function. We will study a wide range of typical spectra in subsequent sections. Additionally, the term, $dW(\eta)\otimes dW(k)$, is complex two-dimensional Gaussian white noise satisfying

$$\langle dW(\eta)\rangle=\langle dW(k)\rangle=0$$

and

$$\langle dW(\eta)\otimes dW(k),dW(\eta')\otimes dW(k')\rangle$$
$$=\delta(\eta+\eta')\delta(k+k')d\eta\, d\eta'\, dk\, dk'.$$

We define the ensemble average over both the random advection and random initial data by $\langle\cdot\rangle_{W,\gamma}=\langle\langle\cdot\rangle_\gamma\rangle_W$.

Consider the following N-dimensional parabolic quantum mechanics problem:

$$\frac{\partial\psi}{\partial t}=\bar{\kappa}\,\Delta_N\psi+V(x,t)\psi,$$
$$\psi|_{t=0}=\psi_0(x).$$

The general solution to this problem is given by the Feynman–Kac formula:[18]

$$\psi(t,x)=E_{\mathbf{B}}\left[\exp\left(\int_0^t V[x+(2\bar{\kappa})^{1/2}\mathbf{B}(s),t-s]ds\right)\times\psi_0[x+(2\bar{\kappa})^{1/2}\mathbf{B}(t)]\right]. \quad (5)$$

Here, the function, $\mathbf{B}(s)$, is a realization of N-dimensional Brownian motion with $\mathbf{B}(0)=0$ and $E_{\mathbf{B}}[\cdot]$ denotes averaging over all paths, $\mathbf{B}$.

Next, following work by one of the authors,[15,16] we recognize the connection between the passive scalar governed by Eq. (1) and a parabolic quantum mechanics problem. Through use of a Fourier representation, we observe that the initial data given in Eq. (3), with $\phi(x,y)$ given in Eq. (4) has the following representation:

$$T|_{t=0}=\int\int e^{2\pi i(x\eta+yk)}[\hat{\hat{T}}(\eta,k)d\eta\, dk + E^{1/2}(\eta,k)dW(\eta)\otimes dW(k)],$$

where $\hat{\hat{T}}(\eta,k)$ is the Fourier transform of the deterministic initial data, $T(x,y)$. Since the equation in (1) is linear, we may write the solution using superposition as

$$T(x,y,t)=\int\int e^{2\pi iyk}\psi(t,x,k,\eta)[\hat{\hat{T}}(\eta,k)d\eta\, dk + E^{1/2}(\eta,k)dW(\eta)\otimes dW(k)],$$

where the function, ψ, satisfies the following parabolic quantum mechanics problem:

$$\frac{\partial\psi}{\partial t}=\bar{\kappa}\psi_{xx}-[2\pi ik\gamma(t)x+\bar{\kappa}4\pi^2k^2]\psi,$$
$$\psi|_{t=0}=e^{2\pi i\eta x}. \tag{6}$$

Applying the Feynman–Kac formula to the problem in (6), we arrive at the following function space integral representation for the function, ψ:

$$\psi(t,x,k,\eta)=e^{-4\pi^2k^2t}E_B\left[\exp\left(-2\pi ik\int_0^t\gamma(t-s)\times[x+(2\bar{\kappa})^{1/2}B(s)]ds\right)\times\exp\{2\pi i\eta[x+(2\bar{\kappa})^{1/2}B(t)]\}\right],$$

for a fixed realization of the random function, $\gamma(t)$.

With this representation formula, we may compute arbitrary moments. Computing the ensemble average over both random advection and random initial data, we have

$$\left\langle\prod_{j=1}^{N}T(t,x_j,y_j)\right\rangle_{\gamma,W}=\int_{R^N}\int_{R^N}e^{2\pi i\mathbf{y}\cdot\mathbf{k}}e^{-4\pi^2\bar{\kappa}|\mathbf{k}|^2t}D\langle\Psi\rangle_\gamma,$$

where the term, Ψ, is given by the following expression:

$$\Psi=E_{\mathbf{B}}\left\{\exp-\left[2\pi i\sum_{j=1}^{N}\left(k_j\int_0^t\gamma(t-s)[x_j+(2\bar{\kappa})^{1/2}B_j(s)]ds+\eta_j[x_j+(2\bar{\kappa})^{1/2}B_j(t)]\right)\right]\right\},$$

and the differential, D, is given by

$$D=\left\langle\prod_{j=1}^{N}\hat{\hat{T}}(\eta_j,k_j)d\eta_j\, dk_j+E^{1/2}(\eta_j,k_j)dW(\eta_j)\otimes dW(k_j)\right\rangle_W. \tag{7}$$

Next, define a quantity, X, as

$$X(t)=\int_0^t\gamma(t-s)\sum_{j=1}^{N}k_j[(x_j+(2\bar{\kappa})^{1/2}B_j(s)]ds.$$

For each fixed realization of Brownian motion, $\mathbf{B}(t)$, this quantity, $X(t)$, is a mean zero, Gaussian random variable. Consequently, the ensemble average over realizations of the random function, $\gamma(t)$, amounts to evaluating the characteristic function of the Gaussian random variable, X. The result is

$$\langle\Psi\rangle_\gamma=E_{\mathbf{B}}\left[\exp\left(-2\pi i\sum_{j=1}^{N}\eta_j[x_j+(2\bar{\kappa})^{1/2}B_j(t)]\right)\times\exp\left(-\frac{4\pi^2}{2}\sum_{i,j=1}^{N}k_ik_j\int_0^t\int_0^t\delta(s-s')[x_j+(2\bar{\kappa})^{1/2}B_j(s)][x_i+(2\bar{\kappa})^{1/2}B_i(s')]ds\, ds'\right)\right].$$

Integrating the delta function gives the following:

$$\langle\Psi\rangle_\gamma=E_{\mathbf{B}}\left[\exp\left(-\frac{4\pi^2}{2}\sum_{i,j=1}^{N}k_ik_j\int_0^t[x_j+(2\bar{\kappa})^{1/2}B_j(s)]\times[x_i+(2\bar{\kappa})^{1/2}B_i(s)]ds-2\pi i\sum_{j=1}^{N}\eta_j[x_j+(2\bar{\kappa})^{1/2}B_j(t)]\right)\right].$$

But this is immediately recognized as the Feynman–Kac formula for a parabolic quantum harmonic oscillator with potential,

$$V_N(\mathbf{x},\mathbf{k})=-2\pi^2(\mathbf{x}\cdot\mathbf{k})^2.$$

Introducing the notation, $\psi_N=\langle\Psi\rangle_\gamma$, we then have that the N-body parabolic quantum wave function, ψ_N, satisfies the following partial differential equation subject to plane wave initial data:

$$\frac{\partial\psi_N}{\partial t}=\bar{\kappa}\,\Delta_N\psi_N-2\pi^2(\mathbf{k}\cdot\mathbf{x})^2\psi_N,$$
$$\psi|_{t=0}=e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{x}}. \tag{8}$$

Consequently, we have the following result for arbitrary moments of the temperature field, T, satisfying Eq. (1):

$$\left\langle\prod_{j=1}^{N}T(x_j,y_j,t)\right\rangle=\int_{R^N}\int_{R^N}e^{2\pi i\mathbf{k}\cdot\mathbf{y}-4\pi^2|\mathbf{k}|^2\bar{\kappa}t}\psi_N(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t)D. \tag{9}$$

The equation in (8) has an exact solution formula that is obtained through an application of Mehler's formula.[19] We present this exact solution here, and defer the derivation to the Appendix.

The solution to Eq. (8) is

$$\psi_N=\bar{f}(b\hat{\mathbf{k}}\cdot\mathbf{x},at,\tilde{\alpha})\bar{g}(\mathbf{x}_{k_\perp},t,\boldsymbol{\eta}_{k_\perp}), \tag{10}$$

where the scalings are given by

$$a=\sqrt{2\pi^2\bar{\kappa}}|\mathbf{k}|, \tag{11}$$

$$b=\left(\frac{2\pi^2}{\bar{\kappa}}\right)^{1/4}\sqrt{|\mathbf{k}|}, \tag{12}$$

and the vectors, $\mathbf{x}_{k_\perp}$, $\boldsymbol{\eta}_{k_\perp}$ denote the respective orthogonal complements of $\mathbf{x}$ and $\boldsymbol{\eta}$ with respect to the vector $\mathbf{k}$. The number, $\tilde{\alpha}$, is given by

$$\tilde{\alpha}=2\pi\left(\frac{\bar{\kappa}}{2\pi^2}\right)^{1/4}\frac{\mathbf{k}\cdot\boldsymbol{\eta}}{|\mathbf{k}|^{3/2}}, \tag{13}$$

and the functions, $\bar{f}$ and $\bar{g}$, are given explicitly as

$$\bar{f}(x,t,\alpha)=[\cosh(2t)]^{-1/2}e^{[(x^2+\alpha^2)/2]\tanh(2t)+ix\alpha\ \mathrm{sech}(2t)}, \tag{14}$$

$$\bar{g}(\mathbf{x},t,\boldsymbol{\alpha})=e^{2\pi i\boldsymbol{\alpha}\cdot\mathbf{x}}e^{-4\pi^2\bar{\kappa}|\boldsymbol{\alpha}|^2t}. \tag{15}$$

Below, we present explicit exact formulas for the first four moments of the temperature field satisfying (1). These formulas are somewhat tedious; however, they are essential in deriving the new phenomena discussed below in Secs. III, IV, and V. The reader not interested in these formulas may skip to Sec. III, and refer back to these formulas when necessary.

A. Exact formulas for the moments, $N=1,2,3,4$

The following formulas are computed using the quantum mechanical representation just presented in the previous section, along with standard cluster expansions. Details of these procedures will be relegated to the Appendix.

With the solutions just presented in Eqs. (10)–(15), we may find simple closed form expressions for the moments of the temperature field, T. We summarize the first four moments here.

We adopt the following notation to emphasize situations when the mean and fluctuating components of the initial temperature profiles are acting in an isolated fashion as opposed to when interaction between these components is present. Let $\langle(\bar{T})^n\rangle$ denote the nth moment of a temperature field whose initial data is the purely deterministic function, $\bar{T}(x,y)$. Further, let $\langle(\Phi)^n\rangle$ denote the nth moment of a temperature field with initial data, which is the fluctuating random field presented in Eq. (4). We will denote the fluctuation of any random field, R, by

$$\delta R=R-\langle R\rangle.$$

1. The mean

The mean of the temperature field is given by

$$\langle T\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2\pi iky-4\pi^2k^2\bar{\kappa}t}\psi_1(x,k,n,t)\hat{\bar{T}}(\eta,k)d\eta\ dk, \tag{16}$$

with

$$\psi_1=\bar{f}[b\ \mathrm{sgn}(k)x,at,\tilde{\alpha}], \tag{17}$$

where the scalings a, b, and $\tilde{\alpha}$ are given explicitly in Eqs. (11)–(13), respectively.

2. The second moment

The second moment of the temperature field is given by

$$\langle T^2\rangle=\langle\bar{T}^2\rangle+\langle\Phi^2\rangle,$$

where

$$\langle\bar{T}^2\rangle=\int_{R^2}\int_{R^2}e^{2\pi i\mathbf{k}\cdot\mathbf{y}-4\pi^2|\mathbf{k}|^2\bar{\kappa}t}\hat{\bar{T}}(\eta_1,k_1)\hat{\bar{T}}(\eta_2,k_2)\times\psi_2(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t), \tag{18}$$

$$\langle\Phi^2\rangle=\int_{R^1}\int_{R^1}d\eta\ dk\ E(\eta,k)e^{-8k^2\pi^2\bar{\kappa}t}\bar{f}(0,at,\tilde{\alpha}). \tag{19}$$

Observe the effects of mean and fluctuating initial data decouple; the second moment being written as a superposition of second moments of temperature fields arising from either purely deterministic or purely fluctuating initial profiles.

Additionally, we may calculate the temperature variance:

$$\langle\delta T^2\rangle=\langle T^2\rangle-\langle T\rangle^2.$$

Relating this expression to the isolated cases of pure mean and pure fluctuation, we see then that

$$\langle\delta T^2\rangle=\langle\delta\bar{T}^2\rangle+\langle\Phi^2\rangle. \tag{20}$$

Again, $\delta\bar{T}$ is the fluctuation of the temperature profile produced by deterministic initial data and Φ is the temperature profile with fluctuating initial data.

3. The third moment

The third moment of the temperature field is given by

$$\langle T^3\rangle=\langle\bar{T}^3\rangle+\langle\Phi^2\bar{T}\rangle^{1,2}. \tag{21}$$

The expression, $\langle\Phi^2\bar{T}\rangle^{1,2}$, represents nontrivial coupling between the mean temperature and the fluctuating profile. It is given by the following:

$$\langle\Phi^2\bar{T}\rangle^{1,2}=3\int_{R^2}d\eta_3\ dk_3\int_{R^2}d\eta\ dk\ e^{2\pi ik_3y-4\pi^2|\mathbf{k}|^2\bar{\kappa}t}\times E(\eta,k)\hat{\bar{T}}(\eta_3,k_3)\psi_3(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t), \tag{22}$$

where the vectors $\mathbf{x}$, $\mathbf{k}$, and $\boldsymbol{\eta}$ are

$$\mathbf{x}={}^t(x,x,x),$$

$$\mathbf{k}={}^t(k,-k,k_3),$$

$$\boldsymbol{\eta}={}^t(\eta,-\eta,\eta_3).$$

Further, the function, ψ_3, is given by the expression in (10), and when evaluated with these vectors yields

$$\psi_3(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t)=\bar{f}\left(b,\frac{k_3}{|\mathbf{k}|}x,at,\tilde{\alpha}\right)\bar{g}(\mathbf{x}_{k_\perp},t,\boldsymbol{\eta}_{k_\perp}). \tag{23}$$

The function $\bar{g}$ has the following simple form:

$$\bar{g}(\mathbf{x}_{k_\perp},t,\boldsymbol{\eta}_{k_\perp})=e^{2\pi i[-(2kx/|\mathbf{k}|^2)(k_3\eta-k\eta_3)^2]}\times e^{-8\pi^2\bar{\kappa}t[(k_3\eta-k\eta_3)^2/|\mathbf{k}|^2]}.$$

With these expressions, we may compute the non-normalized skewness:

$$\langle\delta T^3\rangle=\langle\delta\bar{T}^3\rangle+Q_3, \tag{24}$$

where the coupling term, Q_3, is given explicitly as

$$Q_3=\langle\Phi^2\bar{T}\rangle^{1,2}-3\langle\Phi^2\rangle\langle\bar{T}\rangle. \tag{25}$$

4. The fourth moment

Finally, we present the fourth moment:

$$\langle T^4\rangle=\langle\bar{T}^4\rangle+\langle\Phi^4\rangle+\langle\Phi^2\bar{T}^2\rangle^{2,2}. \tag{26}$$

Here, the last term, $\langle\Phi^2\bar{T}^2\rangle^{2,2}$, is the coupling term given by

$$\langle\Phi^2\bar{T}^2\rangle^{2,2}=6\int_{R^2}d\eta\,dk\int_{R^4}dk_1\,dk_2\,d\eta_1\,d\eta_2\times e^{2\pi i\mathbf{k}\cdot\mathbf{y}-4\pi^2|\mathbf{k}|^2\bar{\kappa}t}\psi_4(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t)E(\eta,k)\times\prod_{j=1}^{2}\hat{\bar{T}}(\eta_j,k_j), \tag{27}$$

where the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{k}$, and $\boldsymbol{\eta}$ are the following:

$$\mathbf{x}={}^t(x,x,x,x),$$

$$\mathbf{y}={}^t(y,y,y,y),$$

$$\mathbf{k}={}^t(k_1,k_2,k,-k),$$

$$\boldsymbol{\eta}={}^t(\eta_1,\eta_2,\eta,-\eta).$$

The term, $\langle\Phi^4\rangle$, in Eq. (26) is given as

$$\langle\Phi^4\rangle=3\int_{R^2}\int_{R^2}d\boldsymbol{\eta}\,d\mathbf{k}\;e^{-4\pi^2(\bar{\kappa}|\mathbf{k}|^2+|\mathbf{z}|^2)t}\bar{f}(0,at,\tilde{\alpha})\times\prod_{j=1}^{2}E(\eta_j,k_j), \tag{28}$$

where the vectors here are given as

$$\mathbf{k}={}^t(k_1,-k_1,k_2,-k_2),$$

$$\boldsymbol{\eta}={}^t(\eta_1,-\eta_1,\eta_2,-\eta_2),$$

$$\mathbf{z}=\boldsymbol{\eta}_{k_\perp}.$$

We remark that the functions, ψ_4 and $\bar{f}$ are given explicitly in Eqs. (10) and (14), respectively. Last, we present an expression for the non-normalized flatness:

$$\langle\delta T^4\rangle=\langle\delta\bar{T}^4\rangle+\langle\Phi^4\rangle+Q_4, \tag{29}$$

and the coupling term is given as

$$Q_4=\langle\Phi^2\bar{T}^2\rangle^{2,2}-12\langle\Phi^2\bar{T}\rangle^{1,2}\langle\bar{T}\rangle+6\langle\Phi^2\rangle\langle\bar{T}^2\rangle. \tag{30}$$

The unusual pairings of wave numbers in the formulas just presented arise from standard cluster expansions. Details may be found in the Appendix.

We comment that the formulas just presented are valid for all time. In subsequent sections, we will explore the long time asymptotic behavior of these four moments, as well as arbitrary moments for cases of isolated deterministic or fluctuating data. We remark that the formulas just presented are useful at finite time as well via elementary numerical quadrature. Further, in light of the exact solution formulas for all moments presented in Eqs. (10)–(15), detailed explicit moment formulas are available for arbitrary N, even for general initial data possessing both mean and fluctuating components. For brevity in exposition, we present only the first four moments here.

III. MEAN INITIAL DATA WITHOUT FLUCTUATION: NONTRIVIAL PERSISTING SKEWNESS

Here, we present simple formulas for the long time behavior of all moments for the special case of purely deterministic initial data, $\bar{T}(x,y)$. In this section, we set $\Phi(x,y)=0$, and assume for simplicity that

$$\hat{\bar{T}}(0,0)=\int\int\bar{T}(x,y)dx\,dy\neq 0.$$

A. Long time asymptotics for arbitrary moments

With these assumptions, we claim that arbitrary moments of the temperature field at long time, $t\to\infty$, are given by the following:

$$\langle T^N\rangle\approx\frac{\bar{A}^N[\hat{\bar{T}}(0,0)]^N}{t^{3N/2}}\int_{R^N}\frac{\sqrt{|\mathbf{k}|}}{\sqrt{\sinh|\mathbf{k}|}}\,d\mathbf{k}, \tag{31}$$

where the constant, $\bar{A}$, is given by

$$\bar{A}=\frac{1}{2^{5/2}\bar{\kappa}\pi^{3/2}}.$$

To verify this claim, recall that the Nth moment of the field given by the equation in (1) is

$$\langle T^N(x,y,t)\rangle=\int_{R^N}\int_{R^N}e^{2\pi i\mathbf{k}\cdot\mathbf{y}}e^{-4\pi^2\bar{\kappa}|\mathbf{k}|^2t}\psi_N(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t)\cdot D, \tag{32}$$

where the N-body quantum wave function, ψ_N, is given in Eqs. (10)–(15) of the previous section, and the differential, D, from (7), takes the simplified form:

$$D=\prod_{j=1}^{N}\hat{\bar{T}}(\eta_j,k_j)d\eta_j\,dk_j.$$

To obtain the large time behavior of the Nth moment, we make the following rescalings:

$$\tilde{\eta}_j=\sqrt{t}\,\eta_j,\quad\tilde{k}_j=\bar{a}tk_j, \tag{33}$$

where the constant, $\bar{a}$, is given by

$$\bar{a}=2^{3/2}\pi\sqrt{\bar{\kappa}}.$$

Making this change of variables, dropping the overtildes, and utilizing the exact formulas for the wave function given in (10)–(15), we find that

$$\langle T^N\rangle = \int_{R^N}\int_{R^N}\bar{D} \cdot (\cosh|\mathbf{k}|)^{-1/2} e^{-c^2(\hat{k}\cdot\boldsymbol{\eta})^2-\beta^2|\boldsymbol{\eta}_{k_\perp}|^2}\tilde{S}(\mathbf{k},\boldsymbol{\eta},\mathbf{x},\mathbf{y},t),$$

where the differential, $\bar{D}$, is given by

$$\bar{D}=\prod_{j=1}^{N}\frac{1}{\bar{a}t^{3/2}}\,\hat{\bar{T}}\left(\frac{\eta_j}{\sqrt{t}},\frac{k_j}{\bar{a}t}\right)d\eta_j\,dk_j. \tag{34}$$

The function, $\tilde{S}$, is a bounded function with the property that

$$\lim_{t\to\infty}\tilde{S}=1.$$

Explicit formulas for the function, $\tilde{S}$, may be found in work by one of the authors.[17] Further, the terms c and β are the following:

$$c^2=\frac{4\pi^2\bar{\kappa}\tanh|\mathbf{k}|}{|\mathbf{k}|},$$
$$\beta^2=4\pi^2\bar{\kappa}.$$

The long time asymptotic state is given by the formula

$$\langle T^N\rangle \approx \left(\frac{\hat{\bar{T}}(0,0)}{\bar{a}t^{3/2}}\right)^N \int_{R^N}\int_{R^N}(\cosh|\mathbf{k}|)^{-1/2} \times e^{-c^2(\hat{k}\cdot\boldsymbol{\eta})^2-\beta^2|\boldsymbol{\eta}_{k_\perp}|^2}\,d\boldsymbol{\eta}\,d\mathbf{k}, \tag{35}$$

provided the expression on the right-hand side of (35) is finite. The N-dimensional integral over the η coordinates may be computed explicitly, with a finite value, establishing the validity of this formula. We defer the details of this calculation to the Appendix. The result of this explicit integration establishes the claim

$$\langle T^N\rangle \approx \frac{1}{t^{3N/2}}\bar{A}^N\int_{R^N}\frac{|\mathbf{k}|d\mathbf{k}}{\sqrt{\sinh|\mathbf{k}|}},$$

where the constant, $\bar{A}$, is the exact constant appearing in (31).

B. Persistence of nontrivial skewness

With the formulas for moments in (31), we demonstrate that the PDF resulting from deterministic initial data need not be symmetric. We do this by showing that the skewness is nonzero and persists for all time. This calculation will illustrate in a simple fashion how the limiting PDF may exhibit long time memory of the initial data.

Define the skewness to be

$$S=\frac{\langle(\delta T)^3\rangle}{\langle(\delta T)^2\rangle^{3/2}},$$

where δT is the fluctuation. Clearly, since

$$\langle(\delta T)^2\rangle=\langle T^2\rangle-\langle T\rangle^2,$$

we have then from the asymptotic moment formula given in Eq. (31) that at long times,

$$\langle(\delta T)^2\rangle\approx\frac{\bar{A}^2\hat{\bar{T}}(0,0)^2}{t^3}\bar{I},$$
$$\bar{I}=\left[\int_{R^2}\omega(\mathbf{k})d\mathbf{k}-\left(\int_{R^1}\omega(k)dk\right)^2\right]. \tag{36}$$

Here, the function, $\omega(\mathbf{k})$, is

$$\omega(\mathbf{k})=\frac{\sqrt{|\mathbf{k}|}}{\sqrt{\sinh|\mathbf{k}|}}. \tag{37}$$

Further, recall that the third moment of the fluctuation is given by

$$\langle(\delta T)^3\rangle=\langle T^3\rangle-3\langle T^2\rangle\langle T\rangle+2\langle T\rangle^3,$$

which, with the formula in Eq. (31), gives

$$\langle(\delta T)^3\rangle\approx\frac{\bar{A}^3\hat{\bar{T}}(0,0)^3}{t^{9/2}}\cdot II, \tag{38}$$

where the constant, II, is given as

$$II=\left[\int_{R^3}\omega(\mathbf{k})d\mathbf{k}-3\int_{R^2}\omega(\mathbf{k})d\mathbf{k}\int_{R^1}\omega(k)dk +2\left(\int_{R^1}\omega(k)dk\right)^3\right]. \tag{39}$$

Performing the required numerical quadratures, we find that

$$\frac{II}{I^{3/2}}\approx 0.757\,416. \tag{40}$$

Thus, we find that the long time limit of the skewness is given by

$$\lim_{t\to\infty}S=\frac{II}{I^{3/2}}\cdot\mathrm{sgn}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{T}(x,y)dx\,dy\right). \tag{41}$$

Here we see the first history effect; namely, the sign of the space average of the initial temperature profile sets the preferred state direction in the limiting distribution function. We remark that such history effects are not often considered in the literature, but clearly may play an important role.

IV. INITIAL RANDOM FLUCTUATION WITHOUT A MEAN

We turn now to situations where the initial temperature field is purely fluctuating. In this section, we set $\bar{T}(x,y)=0$, and choose the energy spectrum, $E(\eta,k)$, to be

$$E(\eta,k)=|k|^{2\alpha}|\hat{\phi}_c(\eta,k)|^2, \tag{42}$$

where the function, $\hat{\phi}_c(\eta,k)$ is a rapidly decreasing function and provides an effective ultraviolet cutoff, with the property that $\hat{\phi}_c(0,0)\neq 0$. The parameter, α, sets the thermal correlation scale. The limiting cases are $\alpha\to-\frac{1}{2}$, which corresponds to a divergence of the large-scale energy in the random initial temperature field, and $\alpha\to+\infty$, which corresponds to a very short range temperature correlation. We suggest the following physical intuition for the behavior of the PDF as the thermal correlation scale is varied. When the thermal correlation scale is comparable with the velocity correlation scale, we expect the PDF to be roughly Gaussian, indicative of central limit-type behavior. However, in the opposite extreme when the thermal correlation scale is very short relative to the velocity correlation scale, we expect the PDF to be highly intermittent with very long tails. Recall that the correlation scale of the velocity field considered in this paper, see Eq. (2), is formally infinite. Consequently, this intuition suggests that the respective limiting cases in the parameter, α, described above yield this picture of intermittency. In this and the next sections, this intuition will be explicitly demonstrated within the context of the model studied here.

A. Formulas for arbitrary moments at long time

Here, we present formulas for arbitrary moments valid at long times. With the assumptions outlined above, we claim that T is a symmetric, mean zero random field whose odd moments vanish and whose even moments at long times are given by

$$\langle T^{2N}\rangle \approx \frac{(2N)!}{2^N(N)!}\frac{\bar{\bar{A}}^{2N}|\hat{\phi}_c(0,0)|^{2N}}{t^{(3/2+2\alpha)N}} \times \int_{R^N}\frac{\sqrt{|\mathbf{k}|}}{\sqrt{\sinh|\mathbf{k}|}}\prod_{j=1}^{N}|k_j|^{2\alpha}\,dk_j, \tag{43}$$

where the constant, $\bar{\bar{A}}$, is given by

$$\bar{\bar{A}} = \frac{1}{2^{7/4+2\alpha}\pi^{3/4+\alpha}\bar{\kappa}^{(1+\alpha)/2}}.$$

Observe that the decay rates for these moments are different from those observed in the previous section for the case of purely deterministic initial data. The different scaling properties of white noise as the energy spectrum varies through the parameter, α, leads to these differences reported here, as compared with the results in Sec. III. This difference suggests the possibility for competition between mean and fluctuating initial data when both effects are present. We will establish in Sec. V that this competition is the mechanism leading to a phase transition in the limiting probability distribution function.

We comment that general moment formulas are readily available for finite time as well. These formulas are quite lengthy, but would only require standard numerical quadrature for interpretation.

It is a straightforward calculation to verify that the quadrature formulas involved in (43) are subadditive, as was established previously for the case of stratified initial data by one of the authors.[15] Consequently, the limiting distribution is non-Gaussian with broad tails, even for the general, nonstratified initial data considered in this paper.

Verifying the formulas presented in Eq. (43) involves a very similar calculation to that of the previous section for deterministic data. The main complication here is that the differential, D, given in (7), requires computing the expectation over the Gaussian random initial data,

$$D = \left\langle \prod_{j=1}^{2N}|k_j|^{\alpha}\hat{\phi}_c(\eta_j,k_j)\,dW(\eta_j)\otimes dW(k_j)\right\rangle_W.$$

This expectation is computed using a straightforward cluster expansion in which averaging is performed in a pairwise fashion. Details of this expansion may be found in the Appendix and in work by one of the authors.[17] The result of this cluster expansion is a simplified version of the general representation formula given in Eq. (9),

$$\langle T^{2N}\rangle = \frac{(2N)!}{2^N(N)!}\int_{R^N}\int_{R^N} e^{2\pi i\mathbf{k}\cdot\mathbf{y}-4\pi^2|\mathbf{k}|^2\bar{\kappa}t}\psi_{2N}(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t) \times\prod_{j=1}^{N}|k_j|^{2\alpha}|\hat{\phi}_c(\eta_j,k_j)|^2\,dk_j\,d\eta_j, \tag{44}$$

where the simplified wave function, ψ_{2N}, is

$$\psi_{2N}(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t) = \bar{f}(0,at,\tilde{\alpha})e^{-4\pi^2\bar{\kappa}|\boldsymbol{\eta}_{k_\perp}|^2 t}. \tag{45}$$

Here, the function, $\bar{f}$, and the quantities, a and $\tilde{\alpha}$, are given previously in Sec. II in Eqs. (14), (11), and (13), respectively. Additionally, the $2N$-dimensional vectors involved in (44) and (45) are the following:

$$\mathbf{y} = (y,y,\ldots,y),$$

$$\mathbf{k} = (k_1,-k_1,k_2,-k_2,\ldots,k_N,-k_N),$$

$$\boldsymbol{\eta} = (\eta_1,-\eta_1,\eta_2,-\eta_2,\ldots,\eta_N,-\eta_N).$$

Observe that the integration is performed over the space $R^N\times R^N$, whereas the vectors, $\boldsymbol{\eta}$, $\mathbf{k}$, as defined above, are elements of the space R^{2N}. To overcome this difficulty, we define new N-dimensional vectors, $\tilde{\boldsymbol{\eta}}$, $\tilde{\mathbf{k}}$, through

$$\tilde{\mathbf{k}} = \sqrt{2}(k_1,k_2,\ldots,k_N),$$

$$\tilde{\boldsymbol{\eta}} = \sqrt{2}(\eta_1,\eta_2,\ldots,\eta_N).$$

Surprisingly, the integrand in Eq. (44) remains invariant upon insertion of these new vectors. Consequently, changing to these new variables requires only bookkeeping the factors of 2. The resulting integral has precisely the same form as the integrals computed in the previous section, and, consequently, the same time rescalings utilized in Sec. III in Eq. (33) will capture the long time behavior. This leads to the formulas in Eq. (43).

B. The long time asymptotic flatness

Using the formulas for moments in (43), we may compute the exact long time limit of the normalized asymptotic flatness. We define the flatness, F, to be

$$F = \frac{\langle(\delta T)^4\rangle}{\langle(\delta T)^2\rangle^2}.$$

The initial data considered in this section leads to a symmetric probability distribution function. Consequently, we have

$$\delta T = T.$$

Then, using the asymptotic moment formulas in (43), we find

$$F_\alpha = 3\frac{\int_{R^2}\omega(\mathbf{k})(|k_1||k_2|)^{2\alpha}\,dk_1\,dk_2}{(\int_{R^1}\omega(k)k^{2\alpha}\,dk)^2}. \tag{46}$$

Table I illustrates how the flatness varies with the parameter α. Recall that the flatness for a Gaussian random variable, as

TABLE I. Asymptotic flatness versus correlation parameter α.

α	F_α
-0.4	3.0168
-0.2	3.1562
0	3.435
0.4	4.455
0.75	6.0083
2	22.839
4	265.02

defined here, is 3. Observe that as α decreases, the flatness approaches 3, suggesting that the distribution tends toward a Gaussian for this limiting case. On the other hand, the flatness becomes unbounded in the opposite extreme as $\alpha \to \infty$. In this situation, the distribution must be much broader than that of a Gaussian. This observation precisely demonstrates the physical intuition outlined above, which relates scalar intermittency with relative magnitudes of thermal and velocity correlation scales.

We remark that the limiting case $\alpha \to \infty$ is formally equivalent to random wave initial data of the form $T_0(y) = a \sin(2\pi k_0 y) + b \cos(2\pi k_0 y)$, with a and b Gaussian random variables. This case was explored previously by one of the authors, and it was established that, for such initial data, the long time limit of the flatness is infinite.[15] Such extreme cases are examples of situations where the thermal correlation scale is much shorter than the velocity correlation scale. The intuition suggested in this section predicts that the resulting PDF will have very long algebraic tails.

V. COMPETING EFFECTS OF MEAN AND FLUCTUATING INITIAL DATA

Next, we explore the situation involving a temperature field that initially has both mean and fluctuating components. First, we will explore the ensuing phenomena ignoring the effects of the deterministic velocity field. Following this, we will consider effects of the mean flow.

Observe that the decay rates for asymptotic moments given in (43) for the case of fluctuating initial data depend upon the correlation parameter α, and are generally different from the decay rates given in (31) for purely deterministic data. These different scalings suggest a potential competition between the effects of mean and fluctuation at long time. Clearly, as $\alpha \to -\frac{1}{2}$, fluctuation dominates, whereas at the opposite extreme, $\alpha \to \infty$, the mean dominates the decay. Additionally, at intermediate α, we expect some sort of balance between these components. We will establish that this competition does lead to a phase transition in the PDF for temperature fluctuations as α is varied. We remind the reader that α sets the correlation scale of fluctuations in the initial temperature field.

In Sec. II we derived the following formulas for the non-normalized skewness and flatness for general initial data possessing both mean and fluctuating components:

$$\langle \delta T^3 \rangle = \langle \delta \bar{T}^3 \rangle + \langle \Phi^2 \bar{T} \rangle^{1,2} - 3\langle \Phi^2 \rangle \langle \bar{T} \rangle, \tag{47}$$

$$\begin{aligned} \langle \delta T^4 \rangle &= \langle \delta \bar{T}^4 \rangle + \langle \Phi^4 \rangle + Q_4, \\ Q_4 &= \langle \Phi^2 \bar{T}^2 \rangle^{2,2} - 12 \langle \Phi^2 \bar{T} \rangle^{1,2} \langle \bar{T} \rangle + 6 \langle \Phi^2 \rangle \langle \bar{T}^2 \rangle. \end{aligned} \tag{48}$$

Recall that the functions $\bar{T}$ and Φ are temperature fields with, respectively, deterministic and fluctuating initial data. Further, the interaction terms $\langle \Phi^2 \bar{T} \rangle^{1,2}$ and $\langle \Phi^2 \bar{T}^2 \rangle^{2,2}$ are given in (22) and (27) of Sec. II.

We have explicit formulas valid at long time for each noninteracting term above in (31) and (43). For fixed values of the correlation parameter α, these formulas demonstrate two distinct decay rates within each of the equations in (47) and (48): one rate being the decay arising from purely deterministic initial data, and the other being the decay rate associated with purely fluctuating data. The decay rate for moments with purely fluctuating data varies with α, and inspection shows that indeed the dominant term does depend upon this correlation parameter. For very large range initial temperature correlations with $\alpha \to -\frac{1}{2}$, fluctuation effects dominate the decay; whereas with short range initial temperature correlations with $\alpha \to \infty$, the decay arising from purely deterministic initial data is dominant at large time. Additionally, we expect to find a critical value of α at which these different decay rates exactly balance. Remarkably, at this transition point, the distribution will depend nontrivially on the Schmidt number.

With these observations in mind, we next present asymptotic formulas for the interaction terms valid at long time. $\langle \Phi^2 \bar{T} \rangle^{1,2}$ satisfies

$$\langle \Phi^2 \bar{T} \rangle^{1,2} \approx \frac{A_{1,2} \hat{\bar{T}}(0,0) |\hat{\phi}_c(0,0)|^2}{t^{3+2\alpha}} \times \int_{R^2} dk_1\, dk_2 |k_1|^{2\alpha} \omega(\mathbf{k}), \tag{49}$$

and the constant $A_{1,2}$ is

$$A_{1,2} = \frac{1}{2^{6+4\alpha} \kappa^{2+\alpha} \pi^{3+2\alpha}}. \tag{50}$$

Similarly, $\langle \Phi^2 \bar{T}^2 \rangle^{2,2}$ satisfies

$$\langle \Phi^2 \bar{T}^2 \rangle^{2,2} \approx \frac{6 A_{2,2} [\hat{\bar{T}}(0,0)]^2 |\hat{\phi}_c(0,0)|^2}{t^{9/2+2\alpha}} \times \int_{R^3} dk_1\, dk_2\, dk_3 |k_1|^{2\alpha} \omega(\mathbf{k}), \tag{51}$$

and the constant, $A_{2,2}$, is

$$A_{2,2} = \frac{3}{2^{17/2+4\alpha} \pi^{9/2+2\alpha} \kappa^{3+\alpha}}. \tag{52}$$

Here, the function, $\omega(\mathbf{k})$ is given in Eq. (37).

We remark that these formulas are derived using cluster expansions and explicit integration over the η coordinates. The derivation follows closely that given previously to obtain long time asymptotic moments for the isolated cases of mean and fluctuating initial data. For complete details, see work by one of the authors.[17]

With the aid of formulas (31), (43), (49), and (51), it is a straightforward calculation to construct explicit expressions for the asymptotic normalized skewness and flatness. This calculation establishes that, at long time, the skewness satisfies

$$S \approx \frac{(C/t^{9/2}+E/t^{3+2\alpha})}{[A/t^3+B/t^{3/2+2\alpha}]^{3/2}}, \tag{53}$$

where A, B, C, and D are the following constants depending on the initial data and the molecular diffusion:

$$A=\frac{\hat{\bar{T}}(0,0)^2}{2^5\bar{\kappa}^2\pi^3}\left[\int_{R^2}\omega(\mathbf{k})d\mathbf{k}-\left(\int_{R^1}\omega(k)dk\right)^2\right],$$

$$B=\frac{|\hat{\phi}_c(0,0)|^2}{2^{7/2+4\alpha}\pi^{3/2+2\alpha}\bar{\kappa}^{1+\alpha}}\int_{R^1}|k|^{2\alpha}\omega(k)dk,$$

$$C=\frac{\hat{\bar{T}}(0,0)^3}{2^{15/2}\bar{\kappa}^3\pi^{9/2}}\, II,$$

$$E=\frac{3\hat{\bar{T}}(0,0)|\hat{\phi}_c(0,0)|^2}{2^{6+4\alpha}\bar{\kappa}^{2+\alpha}\pi^{3+2\alpha}}\left[\int_{R^2}\left|k_1\right|^{2\alpha}\omega(\mathbf{k})dk_1\, dk_2 - \int_{R^1}\omega(k)dk\cdot\int_{R^1}\left|k\right|^{2\alpha}\omega(k)dk\right],$$

and here, the constant, II, is given in (40). We observe that Eq. (53) does exhibit two distinct decay rates, as was alluded to earlier.

Immediately, we may compute the exact long time limit of the skewness. The result is

$$\lim_{t\to\infty} S=\begin{cases} 0, & \text{if } \alpha<\frac{3}{4},\\ S_{3/4}, & \text{if } \alpha=\frac{3}{4},\\ \bar{S}, & \text{if } \alpha>\frac{3}{4}, \end{cases} \tag{54}$$

where $\bar{S}$ is the asymptotic skewness arising from purely deterministic initial data and is given in (41). Interestingly, at the transition point, $\alpha=\frac{3}{4}$, the limiting skewness is seen to depend nontrivially on the Schmidt number,

$$S_{\alpha=3/4}=\frac{[\hat{\bar{T}}(0,0)^3cc+\hat{\bar{T}}(0,0)|\hat{\phi}_c(0,0)|^2dd\bar{\kappa}^{1/4}]}{[\hat{\bar{T}}(0,0)^2aa+|\hat{\phi}_c(0,0)|^2bb\bar{\kappa}^{1/4}]^{3/2}}. \tag{55}$$

The constants aa, bb, cc, and dd involve quadratures. We have evaluated these quadratures numerically and find that $aa=0.266\,713$, $bb=0.502\,392$, $cc=0.104\,331$, and $dd=0.735\,967$.

We now present the long time limit of the flatness. The calculation is very similar to the one just presented. See the work of one of the authors for complete details.[17] The long time limit of the flatness is exactly

$$\lim_{t\to\infty} F=\begin{cases} F_\alpha, & \text{if } \alpha<\frac{3}{4},\\ F_{3/4}, & \text{if } \alpha=\frac{3}{4},\\ \bar{F}, & \text{if } \alpha>\frac{3}{4}. \end{cases} \tag{56}$$

Here, F_α is the flatness arising from purely fluctuating initial data and is given explicitly in (46) and is tabulated in Table I for a wide range of α. Additionally, $\bar{F}$ is the flatness arising from purely deterministic initial data. We compute its value numerically and find $\bar{F}=3.5249$. Similar to the case with the skewness, the constant, $F_{3/4}$, is seen to depend upon the molecular diffusion by

$$F_{3/4}=\frac{\{c_4[\hat{\bar{T}}(0,0)]^4+d_4|\hat{\phi}_c(0,0)|^4\bar{\kappa}^{1/2}+e_4|\hat{\phi}_c(0,0)|^2[\hat{\bar{T}}(0,0)]^2\bar{\kappa}^{1/4}\}}{\{aa[\hat{\bar{T}}(0,0)]^2+bb|\hat{\phi}_c(0,0)|^2\kappa^{1/4}\}^2}. \tag{57}$$

Again, the constants in the above expression involve quadratures. Numerical evaluation yields $c_4=0.250\,75$, $d_4=1.5164$, and $e_4=2.552\,73$.

Formulas (54) and (56) lead to the following interpretation of the phenomena. Namely, for situations where the thermal correlation scale is large, which occurs for $\alpha<\frac{3}{4}$, we see that effects of scalar fluctuations dominate those of scalar mean at long times. In fact, the PDF for temperature fluctuations is seen to approach that established in Sec. IV, where the mean scalar field is absent entirely. At the opposite extreme, for short range correlations in the initial temperature fluctuation with $\alpha>\frac{3}{4}$, deterministic effects of the initial scalar mean dominate at long time. In this case, the distribution is not symmetric, as is evident from the persisting skewness. Further, in this case, the PDF has long tails.

We remark that exact balance occurs for the value of $\alpha=\frac{3}{4}$. At this critical value, a phase transition occurs in the PDF for temperature fluctuations. At this transition point, both the skewness and flatness exhibit dependence upon the molecular diffusion. Additionally, at this critical value, the PDF exhibits memory of both the mean initial temperature profile and the fluctuating cutoff function. Such effects are quite surprising, and the authors are unaware of such examples elsewhere in the literature. We comment that the critical value of $\alpha=\frac{3}{4}$ is a cartoon nicely illustrating the balance of correlations in the temperature field. Within the context of the model studied in this paper, it is readily verified that this is the only transition point for all higher normalized flatness factors.

Finally, we consider the effects of the mean shear flow, which we have ignored until now. In the work of one of the authors,[15] it was established that for stratified Gaussian random initial data the strength of the mean shear flow was key in determining the limiting PDF. To this end, define a positive function, $\mathscr{D}(t)$, by

$$\bar{\mathscr{D}}(t)=\bar{\kappa}\int_0^t\int_0^t\bar{\gamma}(s')\min(s,s')\,\bar{\gamma}(s)ds\;ds'.$$

Then, the limiting PDF will always be Gaussian, provided

$$\lim_{t\to\infty}\frac{\bar{\mathscr{D}}(t)}{t^2}=+\infty. \tag{58}$$

However, if this limit is finite, it was established that the limiting distribution is always broader than Gaussian.[15] We remark that the analogous picture holds for the more general initial data considered in this paper as well; namely, the PDF will be non-Gaussian, provided the limit in (58) is finite. In this case, the limiting distribution exhibits the same phase transition described above. However, if the limit in (58) is infinite, this phase transition no longer persists, as the distribution in this case is Gaussian.

VI. CONCLUDING REMARKS

We have studied an exactly solvable model for a diffusing passive scalar whose initial profile involves both a mean and a fluctuating component. Through analysis of asymptotic moment formulas valid at long time, we have rigorously established that the limiting PDF for temperature fluctuations exhibits a phase transition as the correlation scale of initial temperature fluctuations is varied. Further, we have shown that, provided the thermal correlation scale is large enough, fluctuation effects alone yield the limiting distribution function; whereas for short range correlations in the initial temperature field, memory of the deterministic initial data sets the form of the long time PDF. These effects occur in an interesting fashion for both the skewness and flatness. Additionally, we illustrate that at a critical value of the thermal correlation scale, balance between mean and fluctuating effects occurs, where the PDF is seen to depend nontrivially upon the Schmidt number.

Given the simplicity and robustness of the examples studied here, this model hopefully will provide a stringent test to assess the validity of PDF mapping closures[11] and linear eddy modeling,[9] both at long time, using the asymptotic theory developed here, as well as at finite time using elementary numerical quadratures of the moment formulas developed here.

Finally, we remark that an interesting issue worth pursuing is the assessment of new phenomenon resulting from velocity fields possessing finite correlation times. Such situations are quite complicated, due to the fact that averaging leads to nonlocal equations for which analysis is considerably more difficult. However, we conjecture that at large times, these effects are the same as the results established here in the context of the white noise limit.

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APPENDIX: THE *N* BODY WAVE FUNCTION, THE GAUSSIAN INTEGRAL, AND THE CLUSTER EXPANSION

First, we give details necessary to derive the exact solution formulas given in Eqs. (10)–(15) for the wave function for the quantum harmonic oscillator. Recall that this wave function satisfies the following partial differential equation:

$$\frac{\partial\psi_N}{\partial t}=\bar{\kappa}\;\Delta_N\psi_N-2\pi^2(\mathbf{k}\cdot\mathbf{x})^2\psi_N,$$
$$\psi_{t=0}=e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{x}}. \tag{A1}$$

We diagonalize the N-dimensional potential through an appropriate rotation of the coordinates. Observe that the potential may be written as

$$(\mathbf{k}\cdot\mathbf{x})^2=\mathbf{x}^t(\mathbf{k}\otimes\mathbf{k})\mathbf{x}.$$

Consequently, we arrive at the appropriate transformation by considering the linear algebra problem of diagonalizing the tensor product,

$$\mathbf{k}\otimes\mathbf{k}.$$

Since the tensor product, $\mathbf{k}\otimes\mathbf{k}$, is symmetric, there exists an orthonormal matrix, M, along with a diagonal matrix, D, so that

$$\mathbf{k}\otimes\mathbf{k}=M^t\;DM,$$

where the diagonal matrix, D, is given by

$$D=\begin{pmatrix}|\mathbf{k}|^2 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \cdots\end{pmatrix},$$

and M is an orthonormal matrix given by

$$M=\begin{bmatrix}\hat{\mathbf{k}}\\ \mathbf{k}_\perp\end{bmatrix}, \tag{A2}$$

where $\mathbf{k}_\perp$ denotes the $N-1$-dimensional subspace orthogonal to the unit vector, $\hat{k}$,

$$\hat{\mathbf{k}}=\frac{\mathbf{k}}{|\mathbf{k}|}.$$

Define a transformation by

$$z_i=(M\mathbf{x})_i\,.$$

Then, in the new coordinates, Eq. (59) becomes

$$\frac{\partial\psi}{\partial t}=\bar{\kappa}\;\Delta_N^z\psi-2\pi^2|\mathbf{k}|^2z_1^2\psi,$$
$$\psi|_{t=0}=e^{2\pi i\mathbf{z}\cdot M\boldsymbol{\eta}}. \tag{A3}$$

This equation is immediately solved via separation of variables. Let

$$\psi(\mathbf{z})=A(z_1,t)B(z_2,\ldots,z_N,t).$$

Immediately, it is clear that the function, A, satisfies

$$\frac{\partial A}{\partial t}=\bar{\kappa}\,\frac{\partial^2A}{\partial z_1^2}-2\pi^2|\mathbf{k}|^2z_1^2A,$$
$$A|_{t=0}=e^{2\pi iz_1\hat{\mathbf{k}}\cdot\boldsymbol{\eta}}, \tag{A4}$$

and that the function, B, satisfies

$$\frac{\partial B}{\partial t}=\bar{\kappa}\sum_{j=2}^{N}\frac{\partial^2 B}{\partial z_j^2},$$
$$B|_{t=0}=e^{2\pi i \mathbf{z}_\perp\cdot\boldsymbol{\eta}_{k_\perp}},\tag{A5}$$

where

$$\mathbf{z}_\perp=(0,z_2,z_3,\ldots,z_N),$$

$$\boldsymbol{\eta}_{k_\perp}=\left(I-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\boldsymbol{\eta}.$$

The equation in (A5) is a simple heat equation and is easily solved. Equation (A4) is the equation for a single particle in a harmonic potential that is solved via Mehler's formula.[19] Combining these solutions yields the solution presented in Eqs. (10)–(15).

Next, we present the calculation of the Gaussian integral over the η coordinates given in (35). The integral to evaluate is

$$\overline{II}=\int_{R^N}d\boldsymbol{\eta}\; e^{-c^2(\hat{\mathbf{k}}\cdot\boldsymbol{\eta})^2-\beta^2|\boldsymbol{\eta}_{k_\perp}|^2}.$$

The quadratic form in this integral may be diagonalized with the orthonormal transformation, M, presented above in Eq. (A2). To this end, we make the following change of variables:

$$\mathbf{u}=M\boldsymbol{\eta},$$

then, we have in the new variables,

$$(\hat{\mathbf{k}}\cdot\boldsymbol{\eta})^2=u_1^2,$$

and further, we have

$$|\boldsymbol{\eta}_{k_\perp}|^2=\sum_{j=2}^{N}u_j^2.$$

Using these facts, the N-dimensional integral, $\overline{II}$, becomes

$$\overline{II}=\int_{R^1}du_1\; e^{-c^2u_1^2}\int_{R^{N-1}}\prod_{j=2}^{N}du_j e^{-\beta^2u_j^2}.$$

This is nothing more than the product of N one-dimensional Gaussian integrals. Consequently, we may write

$$\overline{II}=\frac{\pi^{N/2}}{c\beta^{N-1}}.$$

Then, using the expressions for the functions c and β given in Sec. III, we arrive at the sought result.

Finally, we discuss relevant details involved with the cluster expansion of the differential, D, given in (7). First, we discuss the case for purely fluctuating initial data, with $\bar{T}(x,y)=0$. In this case, D takes the following simple form:

$$D=\left\langle\prod_{j=1}^{2N}|k_j|^\alpha\hat{\phi}_c(\eta_j,k_j)dW(\eta_j)\otimes dW(k_j)\right\rangle_W.$$

As a result, the differential may be written as the following sum over expectations of all pairs of Wiener processes appearing above:[16]

$$D=\sum_{\mathbf{i}^-\in P}\prod_{l=1}^{N}\langle dW(k_{i_l^-})\otimes dW(\eta_{i_l^-}),$$
$$dW(k_{i_l^+})\otimes dW(\eta_{i_l^+})\rangle,\tag{A6}$$

where P is the set of all partitions of $2N$ numbers, $\{1,2,\ldots,2N\}$, into N pairs of integers $\{\{i_1^-,i_1^+\},\{i_2^-,i_2^+\},\ldots,\{i_N^-,i_N^+\}\}\in P$. By convention, $i_l^-<i_l^+$, $1\leqslant N$ and $i_k^-<i_{k+1}^-$, $k=1,2,\ldots,N-1$. Further, the number of elements in the set, P, is $(2N)!/2^NN!$.

Using properties of the Wiener process, the expression in (A6) becomes

$$D=\sum_{\mathbf{i}^-\in P}\prod_{l=1}^{N}\delta(k_{i_l^-}+k_{i_l^+})\delta(\eta_{i_l^-}+\eta_{i_l^+})$$
$$\times dk_{i_l^-}\,dk_{i_l^+}\,d\eta_{i_l^-}\,d\eta_{i_l^+}.\tag{A7}$$

Next, consider any element of the sum in (A6). The integral corresponding to this element has the form

$$\int_{R^N}\int_{R^N}e^{2\pi i\mathbf{k}\cdot\mathbf{y}-4\pi^2\bar{\kappa}|\mathbf{k}|^2t}\psi_{2N}(\mathbf{x},\mathbf{k},\boldsymbol{\eta},t)$$
$$\times\prod_{j=1}^{N}|k_j|^{2\alpha}|\hat{\phi}_c(k_j,\eta_j)|^2\,dk_j\,d\eta_j,\tag{A8}$$

where the vectors, $\mathbf{k}$ and $\boldsymbol{\eta}$ have the form

$$\mathbf{k}=(k_1,-k_1,k_2,-k_2,\ldots,k_N,-k_N),$$

$$\boldsymbol{\eta}=(\eta_1,-\eta_1,\eta_2,-\eta_2,\ldots,\eta_N,-\eta_N).$$

We are interested in obtaining results for the moments, $\langle T^N\rangle$. Consequently, we consider the special form of the vectors $\mathbf{x}=(x,x,\ldots,x)$ and $\mathbf{y}=(y,y,\ldots,y)$. With this in mind, it is a straightforward but rather tedious exercise to verify that permuting the vector entries given above leaves the integrand invariant. Consequently, the sum collapses to a single integral multiplied by the total number of permutations.

We remark that similar expansions hold for the case when the mean temperature profile is present. In this case, the expansions involve clustering pairs of the fluctuating components with single mean components. This procedure leads to the unusual pairings of wave vector entries presented in Sec. II. For further details, see the work by one of the authors.[17]

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