Systematic Multiscale Models for the Tropics

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ABSTRACT

Systematic multiscale perturbation theory is utilized to develop self-consistent simplified model equations for the interaction across multiple spatial and/or temporal scales in the Tropics. One of these models involves simplified equations for intraseasonal planetary equatorial synoptic-scale dynamics (IPESD). This model includes the self-consistent quasi-linear interaction of synoptic-scale generalized steady Matsuno–Webster–Gill models with planetary-scale dynamics of equatorial long waves. These new models have the potential for providing self-consistent prognostic and diagnostic models for the intraseasonal tropical oscillation. Other applications of the systematic approach reveal three different balanced weak temperature gradient (WTG) approximations for the Tropics with different regimes of validity in space and time: a synoptic equatorial-scale WTG (SEWTG); a mesoscale equatorial WTG (MEWTG), which reduces to the classical models treated by others; and a new seasonal planetary equatorial WTG (SPEWTG). Both the SPEWTG and MEWTG model equations have solutions with general vertical structure, yet have the linearized dispersion relation of barotropic Rossby waves; thus, these models can play an important role in theories for midlatitude connections with the Tropics. The models are derived both from the equatorial shallow water equations in a simplified context and also as distinguished limits from the compressible primitive equations in general.

1. Introduction

Observational data indicate that through complex interaction of heating and convection, tropical atmospheric flows are organized on a hierarchy of scales ranging from cumulus clouds over a few kilometers to intraseasonal oscillations over planetary scales of order 40 000 km (Nakazawa 1988; Hendon and Salby 1994; Wheeler and Kiladis 1999). Simplified models indicate that such tropical heating patterns interact with the midlatitude flows in the troposphere on both synoptic and seasonal timescales primarily through the generation of and interaction with equivalent barotropic Rossby waves (Kasahara and da Silva Dias 1986; Hoskins and Jin 1991; Wang and Xie 1996). However, many dynamic aspects, including the active role of tropical convection, in this process are not yet understood. Despite substantial theoretical efforts, the mechanisms for the tropical intraseasonal oscillation remain to be elucidated (Raymond 2001, and references therein), while many aspects of convectively coupled tropical waves are poorly resolved in contemporary general circulation models (Slingo et al. 1996). The Matsuno–Webster–Gill models (Matsuno 1966; Webster 1972; Gill 1980) for the linear steady atmospheric response to heating have a prominent role in simplified models for tropical tropospheric flow patterns including interaction with the boundary layer (Lindzen and Nigam 1987; Neelin 1989; Wang and Li 1993). In another direction, exploiting the fact that the horizontal temperature gradients in the equatorial middle troposphere are weak in various flow regimes formally yields simplified equatorial balanced dynamics (Charney 1963; Held and Hoskins 1985; Browning et al. 2000). There has been a recent surge of activity in developing and utilizing such balanced weak temperature gradient (WTG) approximations in the Tropics and subtropics with horizontal advection of moisture included in order to model the regions of surface precipitation and their correlation with middle troposphere moisture and temperature (Sobel et al. 2001; Bretherton and Sobel 2002).

With the observational record and theoretical efforts mentioned above as background, the objective of the present paper is to utilize the tools of systematic multiscale perturbation theory from applied mathematics (Kevorkian and Cole 1981; Majda and Embid 1998;

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Klein 2000) to develop simplified models for capturing the features of multiscale interaction in the Tropics. In particular, this systematic approach reveals three different balanced WTG equations with different regimes of validity in space and time: a synoptic equatorial-scale WTG (SEWTG), a mesoscale equatorial WTG (MEWTG), and a seasonal planetary equatorial WTG (SPEWTG). The MEWTG model yields the classical WTG equations (Charney 1963; Browning et al. 2000; Sobel et al. 2001) while the SEWTG yields in a systematic fashion, modified Gill models for the quasi-stationary response to heat and momentum sources (Gill 1980; Lindzen and Nigam 1987; Neelin 1989) with zonal mean momentum and height corrections. Significantly, the systematic approach also provides new equations that unequivocally determine the temporal evolution of these quasi-steady patterns. To the authors' knowledge, these evolution equations are derived here for the first time. The SPEWTG is a novel model equation that is valid on seasonal timescales and planetary spatial scales. As established below in sections 2 and 3, both the SPEWTG and MEWTG model equations have solutions with general vertical structure yet have the linearized dispersion relations of barotropic Rossby waves; thus, these models can be expected to play an important role in theories for midlatitude connections with the Tropics.

Besides identifying regimes of validity in space and time of various balanced models, more importantly, the systematic approach yields self-consistent model equations for the interaction across multiple space and/or timescales. The most interesting model of this sort developed in this paper involves new simplified equations for intraseasonal planetary equatorial synoptic-scale dynamics (IPESD). These models involve the self-consistent quasi-linear interaction of the SEWTG ''Gill'' models with the planetary-scale dynamics of equatorial long waves (Heckley and Gill 1984). The IPESD equations include both long wavelength Kelvin waves and equatorial Rossby waves in the dynamics and have the potential for providing self-consistent prognostic and diagnostic models for the tropical intraseasonal oscillation. Other applications developed briefly below involving multiple timescales yield equations for the self-consistent interaction of linear equatorial waves with the balanced SEWTG and MEWTG dynamics (for similar results in midlatitude see Majda and Embid 1998; Embid and Majda 1996, 1998).

In section 2, the systematic multiscale methods are introduced and applied to the equatorial shallow water equations to derive conceptually simplified versions of the IPESD, SEWTG, MEWTG, and SPEWTG models in this idealized situation. A self-contained derivation of IPESD through multiple-scale methods is utilized here to motivate the more general applications from section 3. The full power of the systematic multiscale approach is demonstrated in section 3 where appropriate IPESD, SEWTG, MEWTG, and SPEWTG models are derived from the compressible primitive equations on various interacting spatiotemporal timescales through distinguished limits involving a single nondimensional small parameter $\varepsilon \cong 1/8$, which is utilized simultaneously for all the different regimes. This last fact can be anticipated by earlier work for midlatitudes where one of the authors (Klein 2000) demonstrated the systematic emergence of the classical stratified quasi-geostrophic equations on synoptic scales as well as other classical midlatitude dynamic equations from the compressible primitive equations through a single nondimensional small parameter.

2. Systematic multiscale models for equatorial shallow water

The equatorial shallow water equations with momentum and geopotential source terms are given by

$$\frac{D\mathbf{v}}{Dt} + \beta y\mathbf{v}^{\perp} + g\nabla H = S_{\mathbf{v}}$$
$$\frac{DH}{Dt} + H\,\mathrm{div}\mathbf{v} = S_h. \qquad (2.1)$$

In (2.1) the horizontal velocity $\mathbf{v}$ is given by $\mathbf{v} = (u, v)$, with $\mathbf{v}^{\perp} = (-v, u)$ and $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, while H is the geopotential height. Here and elsewhere in the paper, the parameterization of turbulent transport, radiation, moisture, and boundary layer physics are all subsumed in the source terms; thus, it will be an important consideration to assess the allowed strength of the source terms in each asymptotic regime. This will be developed below at the end of this section. Models of this sort arise in the Tropics through Galerkin projection on a dominant vertical baroclinic mode (Gill 1980; Wang and Li 1993; Neelin and Zeng 2000; Majda and Shefter 2001). With a choice of reference length and timescales ℓ_{ref}, t_{ref}, and reference height H_0, the equations in (2.1) can be rewritten in the nondimensional form,

$$\frac{D\mathbf{v}}{Dt} + \tilde{\beta} y\mathbf{v}^{\perp} + F(\mathrm{Fr})^{-2}\nabla h = \tilde{S}_{\mathbf{v}}$$
$$\frac{Dh}{Dt} + h\,\mathrm{div}\mathbf{v} + F^{-1}\,\mathrm{div}\mathbf{v} = \tilde{S}_h, \qquad (2.2)$$

where

$$v_{\mathrm{ref}} = \frac{\ell_{\mathrm{ref}}}{t_{\mathrm{ref}}}, \qquad C_{\mathrm{ref}} = \sqrt{gH_0} \qquad H = H_0(1 + Fh),$$
$$\tilde{\beta} = \beta\ell_{\mathrm{ref}}t_{\mathrm{ref}} \qquad \tilde{S}_{\mathbf{v}} = \frac{t_{\mathrm{ref}}}{v_{\mathrm{ref}}}S_{\mathbf{v}}, \qquad \tilde{S}_h = \frac{t_{\mathrm{ref}}}{H_0F}S_h$$
$$\mathrm{Fr} = \frac{v_{\mathrm{ref}}}{C_{\mathrm{ref}}}, \qquad (2.3)$$

so that Fr is the Froude number, F measures the nondimensional strength of geopotential height perturbations, and $\tilde{\beta}$ can be viewed as the reciprocal of a β-based Rossby number. In all the developments in this section, the representative standard values

$$C_{\mathrm{ref}} = 50 \mathrm{~m~s}^{-1}, \qquad v_{\mathrm{ref}} = 5 \mathrm{~m~s}^{-1}$$
$$\mathrm{Fr} = \varepsilon = 0.1, \qquad F = \varepsilon = 0.1 \tag{2.4}$$

are always utilized (Wang and Li 1993; Majda and Shefter 2001; Bretherton and Sobel 2002) with a variety of multiple space and timescales defining ℓ_{ref}, t_{ref}. Note that geopotential height perturbations of 10% are allowed by the choice of F in (2.4); the discussion below in section 2d shows these values are compatible with tropical heating. The reduced equations are derived below formally for $\varepsilon \ll 1$ and the value $\varepsilon = 0.1$ is utilized as a conservative choice to demonstrate realistic physical effects.

a. *Overview of the scaling regimes*

The standard equatorial synoptic length and timescales are defined by $\ell_s = (C_{\mathrm{ref}}/\beta)^{1/2} = 1500$ km, $T_{\mathrm{ES}} = (C_{\mathrm{ref}}\beta)^{-1/2} = 8$ h; another fundamental length scale is the Charney inertial scale, $\ell_m = (v_{\mathrm{ref}}/\beta)^{1/2}$ with corresponding advective timescale $T_m = \ell_m/v_{\mathrm{ref}}$. It is useful to provide an overview through these fundamental length and timescales of the physical intuition and terminology in the equations derived below.

1) THE IPESD REGIME

Intuitively, one regime of balanced equatorial dynamics occurs on the equatorial synoptic length scale and corresponding advective timescale $T_I = \ell_s/v_{\mathrm{ref}} = \varepsilon^{-1}T_{\mathrm{ES}} \approx 3.33$ days with $\varepsilon = 0.1$; since 10 units of T_I span more than 1 month, T_I is an intraseasonal timescale. The balanced dynamics derived below on these scales are called the synoptic equatorial WTG equations. Since the equatorial circumference is 40 000 km, it is very natural to have modulations of this synoptic-scale behavior on a zonal planetary scale $X_p = \varepsilon x$; with $\varepsilon = 0.1$ and $\ell_s = 1500$ km, the basic length scale for the planetary-scale variation is 15 000 km. The equations derived below, which account for this multiple spatial scale interaction, are the intraseasonal planetary equatorial synoptic-scale dynamics derived below.

2) THE MEWTG AND SPEWTG REGIMES

Intuitively, the Charney inertial length and timescales, ℓ_m and T_m, defined above, give another regime of balanced equatorial dynamics; with $\ell_m = (v_{\mathrm{ref}}/\beta)^{1/2} = (\mathrm{Fr})^{1/2}\ell_s$ and $\varepsilon = 0.1$ in (2.4), $\ell_m = 500$ km is an equatorial mesoscale while T_m is roughly 1.1 days. The mesoscale equatorial WTG equations are the terminology for the resulting balanced dynamics derived below with these reference scales. Balanced dynamics near the equator often involves zonal variations of physical quantities on a larger length scale compared with meriodonal variations. Assume meridional variations are measured by the Charney inertial scale ℓ_m, but zonal variationals occur on the larger length scale $\ell_{\mathrm{sp}} = \varepsilon^{-1}\ell_m$; for $\varepsilon = 0.1$, ℓ_{sp} is 5000 km. Assume that time is measured by the zonal advection timescale $T_s = \ell_{\mathrm{sp}}/v_{\mathrm{ref}}$; for $\ell_{\mathrm{sp}} = 5000$ km, T_s is 11.1 days and 10 units of this timescale cover a season. The suitable balanced dynamics derived below which emerge with these fundamental scales are called the seasonal planetary equatorial WTG equations.

b. *Systematic derivation of the IPESD models*

With the choices $\ell_{\mathrm{ref}} = \ell_s$, $t_{\mathrm{ref}} = T_I$ and (2.4), the nondimensional shallow water equations in (2.2) are given by

$$\frac{D\mathbf{v}}{Dt} + \varepsilon^{-1}(y\mathbf{v}^{\perp} + \nabla h) = \varepsilon^{-1}\hat{S}_{\mathbf{v}}$$
$$\frac{Dh}{Dt} + h\,\mathrm{div}\mathbf{v} + \varepsilon^{-1}\,\mathrm{div}h = \varepsilon^{-1}\hat{S}_h, \quad \text{with}$$
$$\varepsilon^{-1}\hat{S}_{\mathbf{v}} = \tilde{S}_{\mathbf{v}}, \qquad \varepsilon^{-1}\hat{S}_h = \tilde{S}_h. \tag{2.5}$$

Systematic asymptotic solutions of (2.5) will be developed below which, besides varying on the equatorial synoptic scale in (x, y), also vary on the zonal planetary scale $X_P = \varepsilon x$.

To derive the IPESD models, solutions of (2.5) for $\varepsilon \ll 1$ are expanded according to the single time-multiple space-scale approach (Kevorkian and Cole 1981; Klein 2000; Majda 2002):

$$\begin{aligned}\mathbf{v} &= \mathbf{V}^{(0)}(\varepsilon x, y, t) + \mathbf{v}^{(0)}(\varepsilon x, x, y, t)\\ &\quad + \varepsilon[\mathbf{v}^{(1)}(\varepsilon x, x, y, t)] + O(\varepsilon^2)\\ h &= H^{(0)}(\varepsilon x, y, t) + h^{(0)}(\varepsilon x, x, y, t)\\ &\quad + \varepsilon h^{(1)}(\varepsilon x, x, y, t) + O(\varepsilon^2),\end{aligned} \tag{2.6}$$

with $\mathbf{V}^{(0)}(X_P, y, t)$, $H^{(0)}(X_P, y, t)$, etc., regarded as functions of the independent variable X_P so that by the chain rule, for any function $g(\varepsilon x, x, y, t)$, $\partial g/\partial x = \varepsilon(\partial g/\partial X_P) + (\partial g/\partial x)$. Here and below, the zonal average over the synoptic scale alone of a general function $g(\varepsilon x, x, y, t)$ is defined by

$$\overline{g}(X_P, y, t) = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} g(X_P, x, y, t)\,dx, \tag{2.7}$$

where L is the characteristic zonal averaging scale. In (2.6), the leading order terms are split so that $\overline{\mathbf{v}^{(0)}} = 0$, $\overline{h^{(0)}} = 0$. The crucial requirements needed to formally guarantee that the terms $\mathbf{V}^{(0)} + \mathbf{v}^{(0)}$, $H^{(0)} + h^{(0)}$ validly describe the leading-order behavior, are the sublinear growth conditions (Kevorkian and Cole 1981; Majda 2002) for the next order terms, $\mathbf{v}^{(1)}$, $h^{(1)}$,

$$\lim_{\varepsilon\to 0}\left(\frac{\mathbf{v}^{(1)}(X_P, X_P/\varepsilon, y, t)}{|X_P/\varepsilon| + 1}\right) = \lim_{\varepsilon\to 0}\left(\frac{h^{(1)}(X_P, X_P/\varepsilon, y, t)}{|X_P/\varepsilon| + 1}\right) = 0. \tag{2.8}$$

When the conditions in (2.8) are violated, the terms $\varepsilon v^{(1)}$, $\varepsilon h^{(1)}$, in (2.6) formally have the same magnitude as $v^{(0)}$, $h^{(0)}$ when $|x| = O(\varepsilon^{-1})$ so that $v^{(0)}$, $h^{(0)}$ alone cannot describe the leading-order behavior. Assume that the source terms have the expansion

$$\begin{aligned}\hat{S}_{\mathbf{v}} &= \hat{S}_{\mathbf{v}}^{(0)} + \varepsilon\hat{S}_{\mathbf{v}}^{(1)} + O(\varepsilon^2)\\ \hat{S}_h &= \hat{S}_h^{(0)} + \varepsilon\hat{S}_h^{(1)} + O(\varepsilon^2),\end{aligned} \tag{2.9}$$

then the systematic multiple-scale procedure sketched below yields the IPESD model equations:

1) $V^{(0)}$ satisfies

$$-yV^{(0)} = \overline{\hat{S}_u^{(0)}} \qquad V_y^{(0)} = \overline{\hat{S}_h^{(0)}}; \tag{2.10}$$

2) $\mathbf{v}^{(0)}$, $h^{(0)}$ with $\overline{\mathbf{v}^{(0)}} = 0$, $\overline{h^{(0)}} = 0$ are uniquely determined by the equatorial synoptic-scale equations

$$\begin{aligned}y(\mathbf{v}^{(0)})^\perp + \nabla h^{(0)} &= \hat{S}_{\mathbf{v}}^{(0)} - \overline{\hat{S}_{\mathbf{v}}^{(0)}}\\ \operatorname{div}\mathbf{v}^{(0)} &= \hat{S}_h^{(0)} - \overline{\hat{S}_h^{(0)}},\end{aligned} \tag{2.11}$$

where the planetary-scale X_P and time enter only as parameters; and

3) $[U^{(0)}(X_P, y, t), H^{(0)}(X_P, y, t), V^{(1)}(X_P, y, t)]$ are determined by the quasi-linear equatorial longwave equations (QLELWE)

$$\begin{aligned}&U_t^{(0)} - yV^{(1)} + V^{(0)}U_y^{(0)} + \overline{v^{(0)}u_y^{(0)}} + H_{X_P}^{(0)} = \overline{\hat{S}_u^{(1)}}\\ &H_t^{(0)} + V^{(0)}H_y^{(0)} + \overline{(v^{(0)}h_y^{(0)})} + (U_{X_P}^{(0)} + V_y^{(1)}) = \overline{\hat{S}_h^{(1)}}\\ &yU^{(0)} + H_y^{(0)} = \overline{\hat{S}_v^{(0)}}.\end{aligned} \tag{2.12}$$

The meridional velocity $V^{(1)}$ in (2.12) is defined as $V^{(1)} = \overline{v^{(1)}}$ from the ansatz in (2.6). Clearly there is a solution $V^{(0)}$ of (2.10) only if $\overline{\hat{S}_u^{(0)}}$ and $\overline{\hat{S}_h^{(0)}}$ satisfy $\overline{\hat{S}_u^{(0)}} = -y\int_0^y \overline{\hat{S}_h^{(0)}} + Ay$ with A an arbitrary constant. The equations in (2.11) express Sverdrup balance and readily have explicit unique solutions with $\overline{\mathbf{v}^{(0)}} = 0$, $\overline{h^{(0)}} = 0$ with the specified source terms (see section 3 below and Bretherton and Sobel 2002); these solutions have the structure of the nondissipative Gill model (Gill 1980) with the important additional fact that both time and the planetary scale enter as parameters. The equations in (2.12) are the linear equatorial longwave equations (Heckley and Gill 1984) with explicit source terms and new coupled effects involving the explicit Reynold's stresses of the synoptic scales, $\overline{v^{(0)}u_y^{(0)}}$ and $\overline{(v^{(0)}h_y^{(0)})}$, as well as meridional advection by $V^{(0)}$; these equations have both Kelvin waves and long wavelength equatorial Rossby waves and are readily solved explicitly with known source terms. Explicit solutions of (2.10)–(2.12) revealing equatorial synoptic-scale and planetary-scale wave interaction as well as readily computed numerical solutions with boundary layer momentum and moisture closure will be presented elsewhere by the authors in the near future. When variation on the planetary scale is ignored in all the variables in (2.10), (2.11), and (2.12)—that is,

$$U^{(0)}(y, t), \qquad V^{(1)}(y, t), \qquad H^{(0)}(y, t) \tag{2.13}$$

—the SEWTG model equations emerge. The SEWTG model is the special case of (2.11), (2.12) where $\mathbf{v}^{(0)}$, $h^{(0)}$ are independent of the planetary scale X_P. Note that the SEWTG equations derived here involve the rectification of (2.11) by leading order corrections $U^{(0)}(y, t)$, $H^{(0)}(y, t)$, which have nonvanishing zonal averages. Importantly, these equations unequivocally determine the temporal evolution of the quasi-steady structures given by Gill-type steady models. Such effects and their role were ignored by Bretherton and Sobel (2002) in their recent study of synoptic equatorial-scale WTG approximations as well as the possibility of planetary-scale wave interaction developed here in (2.11) and (2.12), in general.

The technical motivation for the asymptotic derivation of (2.10), (2.11), and (2.12) begins with the requirement of finding solutions of the equations

$$\begin{aligned}-yv + \partial_x h &= S_u,\\ yu + \partial_y h &= S_v, \quad \text{and}\\ u_x + v_y &= S_h,\end{aligned} \tag{2.14}$$

which satisfy the sublinear growth conditions in (2.8) when X_P and t are regarded as parameters. Furthermore, for illustration, consider the special case of (2.14) with $S_h = 0$ so that $u = -\psi_y$, $v = \psi_x$, and the first equation in (2.14) becomes

$$\frac{\partial}{\partial x}(-y\psi + h) = S_u. \tag{2.15}$$

Clearly (2.15) has a solution that does not grow sublinearly in x if and only if $\overline{S_u} = 0$. With the solvability conditions for (2.14) just discussed above, the derivation of (2.10), (2.11), and (2.12) is straightforward. With the above remarks, inserting (2.6) into (2.5), collecting the terms of order ε^{-1}, and splitting them into fluctuating (zero average in x) and mean terms yields (2.10), (2.11), and the final equation in (2.12). Collecting the terms of order ε^0 yields inhomogeneous equations for $\mathbf{v}^{(1)}$, $h^{(1)}$ of the form in (2.14) where the source terms depend on $U^{(0)}$, $H^{(0)}$, $u^{(0)}$, $h^{(0)}$, $V^{(0)}$, and the longer-scale derivatives, $U_{X_P}^{(0)}$, $H_{X_P}^{(0)}$. Enforcing the solvability condition described below (2.15) for the u-momentum equation yields the first equation in (2.12) with a similar derivation for the second equation in (2.12). This completes the sketch of the derivation. The systematic nature of the asymptotic derivation creates confidence in applying the reduced equations for fixed moderately small values of ε such as $\varepsilon \cong 0.1$ or even somewhat larger values as needed. This general fact is confirmed in a wide variety of applications of multiple-scale techniques used to develop simplified models in diverse physical contexts (Bourlioux and Majda 2000; Klein and Knio 1995; Majda and Bertozzi 2002).

Multiple timescale interaction for equatorial synoptic scales

Consider asymptotic solutions of (2.5) that vary on two timescales, the intraseasonal scale T_I, defining the time unit in (2.5), and the shorter equatorial synoptic timescale $T_{\mathrm{ES}} = \varepsilon T_I$, with spatial fluctuations on the equatorial synoptic scale. In other words, assume that, in contrast to the earlier *single time–multiple length* expansions, $\mathbf{v}$ and h in (2.5) now have the *multiple time–single space* representations:

$$\mathbf{v} = \mathbf{v}^{(0)}(x, y, t, t/\varepsilon) + \varepsilon\mathbf{v}^{(1)}(x, y, t, t/\varepsilon) + O(\varepsilon^2),$$
$$h = h^{(0)}(x, y, t, t/\varepsilon) + \varepsilon h^{(1)}(x, y, t, t/\varepsilon) + O(\varepsilon^2), \tag{2.16}$$

where $\mathbf{v}^{(0)}(x, y, t, T_{\mathrm{ES}})$, $h^{(0)}(x, y, t, T_{\mathrm{ES}})$ are regarded as functions of the independent variables t and T_{ES} in the multiple scaling procedure. First consider the special case of (2.16) where there is only variation in time on the equatorial synoptic scale; then inserting (2.16) into (2.5) and collecting all the terms of order ε^{-1} yields the linear equatorial wave equation:

$$\frac{\partial \mathbf{v}^{(0)}}{\partial T_{\mathrm{ES}}} + y\mathbf{v}^{(0)\perp} + \nabla h^{(0)} = \hat{S}^{(0)}_{\mathbf{v}}$$
$$\frac{\partial h^{(0)}}{\partial T_{\mathrm{ES}}} + \operatorname{div}\mathbf{v}^{(0)} = \hat{S}^{(0)}_{h}, \tag{2.17}$$

so that standard synoptic-scale linear equatorial wave dynamics is recovered on this shorter timescale. For the more general solutions with two timescales in (2.16), equations in (2.5) exactly satisfy the general framework of systematic two timescale averaging theories for geophysical flows [see Eq. (16) of Majda and Embid 1998; Embid and Majda 1996; Majda 2002] so that simplified equations emerge for the interaction of dynamics at the two scales. This is mentioned here to illustrate another use of the multiple-scale formalism and will be developed elsewhere in detail. Similar multiple timescale effects on both equatorial synoptic and mesoscales emerge from the compressible primitive equations in section 3.

c. The MEWTG and SPEWTG models

The Charney inertial length scale and corresponding advective timescale defined above satisfy

$$\ell_m/T_m = \upsilon_{\mathrm{ref}} = 5\ \mathrm{m\ s^{-1}}, \qquad \ell_m T_m \beta = 1. \tag{2.18}$$

With these reference scales and $\mathrm{Fr} = \varepsilon$, $F = \varepsilon$ as in (2.4), the nondimensional shallow water equations in (2.2) have the form

$$\frac{D\mathbf{v}}{Dt} + y\mathbf{v}^{\perp} + \varepsilon^{-1}\nabla h = \varepsilon^{-1}\hat{S}_{\mathbf{v}}, \quad \text{and}$$
$$\frac{Dh}{Dt} + h\operatorname{div}\mathbf{v} + \varepsilon^{-1}\operatorname{div}\mathbf{v} = \varepsilon^{-1}\hat{S}_{h}, \tag{2.19}$$

where the convention below (2.5) has been utilized to redefine the source terms. Assume that the source terms have the expansion in (2.9), while the variables in (2.19) are expanded as a regular perturbation, $\mathbf{v} = \mathbf{v}^{(0)}(x, y, t) + \varepsilon\mathbf{v}^{(1)}$, etc. The leading-order height equation in (2.19) yields the WTG approximation

$$\operatorname{div}\mathbf{v}^{(0)} = \hat{S}^{(0)}_{h}, \tag{2.20}$$

while the curl of the leading-order momentum equation in (2.19) yields

$$\frac{\partial}{\partial t}Q^{(0)} + \operatorname{div}(\mathbf{v}^{(0)}Q^{(0)}) = \varepsilon^{-1}\operatorname{curl}\hat{S}^{(0)}_{\mathbf{v}}, \tag{2.21}$$

with the potential vorticity (PV) $Q^{(0)}$ given by

$$Q^{(0)} = \omega + \beta y, \qquad \omega = -u^{(0)}_{y} + \mathbf{v}^{(0)}_{x}. \tag{2.22}$$

The equations in (2.20), (2.21), and (2.22) are the WTG equations for equatorial shallow water (Sobel et al. 2001). The form of the equations in (2.21) makes it transparent that the linear free waves for this equation have the dispersion relation of barotropic Rossby waves even though these waves have the vertical structure of a baroclinic mode. Such facts are generalized to the compressible equatorial primitive equations in section 3. The above derivation indicates that the standard WTG approximation arises on equatorial mesoscales spatially with a temporal validity of roughly 10 days.

Derivation of SPEWTG models

The SPEWTG models emerge through the basic scalings in (2.18), (2.19), and (2.4) by considering solutions of (2.19) on the unit seasonal timescale $T_S = \varepsilon^{-1}(T_m) = 11.1$ days, with zonal variation on the planetary scale $\ell_{\mathrm{sp}} = \varepsilon^{-1}\ell_m = 5000$ km and with weaker meridional flow and height perturbations so that $\upsilon = \varepsilon\upsilon^{(1)}$ and $h = \varepsilon h^{(1)}$. Thus, asymptotic solutions of (2.19) are sought through the ansatz

$$\mathbf{v} = \begin{pmatrix} u^{(0)}(\varepsilon x, y, \varepsilon t) \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon u^{(1)}(\varepsilon x, y, \varepsilon t) \\ \varepsilon\upsilon^{(1)}(\varepsilon x, y, \varepsilon t) \end{pmatrix} + O(\varepsilon^2),$$
$$h = \varepsilon h^{(1)}(\varepsilon x, y, \varepsilon t) + O(\varepsilon^2), \tag{2.23}$$

where $u^{(0)}(\mathbf{X}_{\mathrm{sp}}, y, T_S)$, $\upsilon^{(1)}(\mathbf{X}_{\mathrm{sp}}, y, T_S)$, and $h^{(1)}(\mathbf{X}_{\mathrm{sp}}, y, T_S)$ are functions of the seasonal time-scale T_S and planetary spatial scale through $\mathbf{X}_{\mathrm{sp}}$. Inserting (2.23) into

(2.19) and collecting the leading-order terms yields the SPEWTG equations:

$$\frac{Du^{(0)}}{Dt} - yv^{(1)} + \frac{\partial h^{(1)}}{\partial \mathbf{X}_{\mathrm{sp}}} = \hat{S}_u^{(2)},$$

$$yu^{(0)} + \frac{\partial h^{(1)}}{\partial y} = \hat{S}_v^{(1)}, \quad \text{and}$$

$$\frac{\partial u^{(0)}}{\partial \mathbf{X}_{\mathrm{sp}}} + \frac{\partial v^{(1)}}{\partial y} = \hat{S}_h^{(1)}, \tag{2.24}$$

where $D/Dt = (\partial/\partial T_S) + u^{(0)}(\partial/\partial \mathbf{X}_{\mathrm{sp}}) + v^{(1)}(\partial/\partial y)$. The third equation in (2.24) is a WTG approximation while the second equation in (2.24) expresses meridonal geostrophic balance. As in (2.22), the curl of the two momentum equations in (2.24) yields the conservation law

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{Q}u^{(0)}}{\partial \mathbf{X}_{\mathrm{sp}}} + \frac{\partial \tilde{Q}v^{(1)}}{\partial y} = 0, \tag{2.25}$$

with the anisotropic potential vorticity

$$\tilde{Q} = -\frac{\partial u^0}{\partial y} + y. \tag{2.26}$$

From (2.25) and (2.26) it follows that linearized free wave solutions of (2.24) have the dispersion relation for long wavelength barotropic Rossby waves—that is, $\omega(k, \ell) = -\ell/k^2$—even though they have the vertical structure of a baroclinic mode. Thus, there is the possibility of resonance with barotropic midlatitude planetary waves through nonlinear advective coupling (Majda and Biello 2002, manuscript submitted to *J. Atmos. Sci.*). This fact is generalized in section 3 for SPEWTG approximations with arbitrary vertical structure. The SPEWTG equations have been derived here under similar anisotropic scalings as the Prandtl boundary layer equations of ordinary incompressible fluid flow (Chorin and Marsden 1990). Here the equatorial waveguide plays the role of the boundary layer.

d. *Strength of the source terms in the simplified models*

The standard dimensional equatorial shallow water equations representing the first baroclinic vertical mode with potential temperature variations (Majda and Shefter 2001) are given by

$$\frac{D\mathbf{v}}{Dt} + \beta y\mathbf{v}^{\perp} - \overline{\overline{\alpha}}\nabla\theta = S_{\mathrm{v}}$$

$$\frac{D\theta}{Dt} - \theta\, \mathrm{div}\mathbf{v} - \overline{\alpha}\, \mathrm{div}\mathbf{v} = S_\theta, \tag{2.27}$$

where

$$\overline{\alpha}\,\overline{\overline{\alpha}} = C_{\mathrm{ref}}^2, \qquad \overline{\alpha} = \frac{H_m(N^2\theta_0)}{g},$$

$$\theta_0 = 300° \text{ K}, \qquad H_m = 5 \text{ km},$$

$$N = 10^{-2} \text{ s}^{-1}, \quad \text{and} \quad C_{\mathrm{ref}} = 50 \text{ m s}^{-1}. \tag{2.28}$$

The corresponding dimensional shallow water equation in (2.1) has the height determined by $h = (-\overline{\overline{\alpha}}/g)\theta$. With the nondimensionalizations $\hat{\theta} = (\theta/\theta_0)$, it follows from (2.28) that

$$\frac{h}{H_0} = 20\frac{\theta}{\theta_0} = 20\hat{\theta} = 2\varepsilon^{-1}\hat{\theta} \quad \text{for } \varepsilon = 0.1. \tag{2.29}$$

From (2.4) $F = \varepsilon = 0.1$ in the IPESD, SEWTG, MEWTG models developed here so that temperature fluctuations $\hat{\theta} = O(\varepsilon^2)$ are required for self-consistency in these models; that is, the equatorial middle troposphere horizontal temperature perturbations can vary 1.5 K to 3 K in accordance with current observational estimates. This temperature scaling is the basis for the results developed in section 3 below. For the SPEWTG equations, which involve averaging over larger spatial scales, from (2.23), weaker temperature perturbations of order 0.15 K to 0.3 K are allowed. To assess the allowed strength of the thermal damping and other thermal temperature sources and sinks, assume for demonstration, the simplified linear relaxation model for thermal damping in the shallow water equation in (2.27)

$$S_\theta = -d_\theta(\theta - \overline{\theta}(x, y, t)) := -d_\theta\tilde{\theta}, \tag{2.30}$$

where $\overline{\theta}$ is an appropriate average potential temperature. Working through the scalings for the nondimensional source terms $\hat{S}_h$ in IPESD or SEWTG yields the formula

$$O(1) = |\hat{S}_h| = (T_{\mathrm{ES}}d_\theta)20\frac{|\tilde{\theta}|}{\theta_0}\varepsilon^{-1}, \tag{2.31}$$

which is satisfied for $\varepsilon = 0.1$, provided the fluctuations $\tilde{\theta}$ roughly satisfy the upper bound

$$(T_{\mathrm{ES}}d_\theta)^{-1}(1.5 \text{ K}) \geq |\tilde{\theta}|. \tag{2.32}$$

In particular, for realistic thermal damping rates of $d_\theta = 0.1$ day^{-1} or less, the condition in (2.32) is easily satisfied self-consistently with $|\tilde{\theta}| \geq 1.5$ K or much larger allowed. These results even remain true for the unrealistically strong thermal damping of the original Gill model where $d_\theta = 0.5$ day^{-1}. These bounds are pessimistic upper bounds for MEWTG, which has a shorter nondimensional time unit, for the overall dynamics. For SPEWTG, the allowed variation of $\tilde{\theta}$ is reduced by the factor 0.1 in a self-consistent manner.

For momentum sources $\hat{S}_{\mathrm{v}}$, arising from boundary layer drag—that is,

$$S_{\mathrm{v}} = -d_{\mathrm{v}}\mathbf{v} \tag{2.33}$$

—it follows that for the IPESD and SEWTG,

$$|\hat{S}_{\mathrm{v}}| = O(T_{\mathrm{ES}}d_{\mathrm{v}}), \tag{2.34}$$

so that roughly the requirement $T_{\mathrm{ES}}d_{\mathrm{v}} \leq 1$ is needed for self-consistency of the asymptotic models. Assuming that d_{v} has the standard aerodynamic drag formula,

$$\frac{C_D^0|v_{\rm ref}|}{h} = d_{\rm v}, \quad C_D^0 = 10^{-3}, \quad h = 500 \text{ m}, \qquad (2.35)$$

yields the value $T_{\rm ES}d_{\rm v} = 0.288$ so that such strong boundary layer drag effects are readily accommodated in the models. As before, the estimate in (2.34) improves even further for MEWTG but more stringent conditions on the forcing are needed for SPEWTG consistent with (2.24). The conditions in (2.32) and (2.34) suggest that for IPESD in some flow regimes and with weaker realistic radiative damping, it might be suitable to utilize $|\hat{S}_h| = O(\varepsilon)$ and $|\hat{S}_{\rm v}| = O(\varepsilon)$ in the theories since $T_{\rm ES}d_v = 2.88\ \varepsilon$ with $\varepsilon = 0.1$.

Energy principles and PV dynamics

Without synoptic-scale source terms, the IPESD equations reduce to the linear equatorial longwave equations. These equations have conservation of a total energy budget, consisting of the sum of zonal kinetic energy and potential energy due to geopotential height perturbations; these equations also conserve the linearized potential vorticity $q = y - u_u - h$, which includes geopotential fluctuations. Thus, the IPESD equations have large-scale conservation principles for both energy and PV, which also incorporate height fluctuations with additional sources and sinks due to equatorial synoptic-scale sources. On the other hand, as shown above in (2.21) and (2.25), both the MEWTG and SPEWTG equations without sources and sinks have conservation of absolute vorticity as in a nondivergent barotropic model and corresponding conservation of total and zonal kinetic energy, respectively, as the basic energy principle. Thus, the MEWTG and SPEWTG models do not incorporate any effects of geopotential height perturbations in either basic energy or PV principles except through sources and sinks. Similar remarks apply for all the corresponding equations with stratification in section 3.

3. Multiple scales for compressible near-equatorial flows

This section carries three central messages.

- In the context of three-dimensional near-equatorial flows described by the compressible primitive equations, there are immediate analogues for all the simplified models derived from the shallow water equations in section 2.
- These analogues are derived here again using systematic multiple-scale asymptotics.
- The derivations in section 2 do not emphasize a systematic relation between the various flow regimes. Thus the reader might question whether these regimes represent mutually exclusive asymptotic limits, or whether the associated flow phenomena may coexist. The present section resolves this issue by showing that all the regimes from section 2 emerge from the 3D compressible flow equations under one and the same set of distinguished limits for the Mach, Froude, and Rossby numbers. The various regimes differ only with respect to their characteristic length and time scalings. Thus these regimes are mutually compatible, and the associated phenomena may be expected to occur simultaneously.

TABLE 1. Bulk microscale reference quantities.

Term	Reference units
Density	$\rho_{\rm ref} = 1$ kg m^{-3}
Pressure	$p_{\rm ref} = 10^5$ kg ms^{-2}
Liquid water mass fraction	$q_{\rm ref} = 10$ g kg^{-1}
Length	$\ell_{\rm ref} = 10$ km ($\sim h_{\rm scale}$)
Velocity	$v_{\rm ref} = 5$ m s^{-1}
Time	$t_{\rm ref} = 30$ min ($\sim \ell_{\rm ref}/v_{\rm ref}$)

The distinguished limits enabling such a unified representation are

$$\sqrt{M} \sim \sqrt{\overline{\rm Fr}} \sim {\rm Ro} := \varepsilon \ll 1, \qquad (3.1)$$

where M, $\overline{\rm Fr}$, and Ro are the Mach, barotropic Froude, and synoptic-scale Rossby numbers, respectively. Note that the barotropic Froude number in (3.1) should not be identified with the gravity wave speed, which arises from potential temperature perturbations and is much smaller (see Table 2 and (3.9) and (3.46) below).

a. Nondimensionalized governing equations

Here, generalized versions of the model equations from section 2 are derived by multiple-scale asymptotics applied to the equations for three-dimensional compressible flow in the equatorial "β plane." Using the reference quantities in Table 1, the nondimensional set of governing equations are

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + w\mathbf{u}_z + \frac{1}{{\rm Ro}_B}(\mathbf{f}\times\mathbf{v})_\parallel + \frac{1}{M^2}\frac{1}{\rho}\nabla p = \mathbf{D}_u,$$

$$w_t + \mathbf{u}\cdot\nabla w + ww_z + \frac{1}{{\rm Ro}_B}(\mathbf{f}\times\mathbf{v})_\perp + \frac{1}{M^2}\frac{1}{\rho}p_z = D_w - \frac{1}{\overline{\rm Fr}^2},$$

$$p_t + \mathbf{u}\cdot\nabla p + wp_z + \gamma p(\nabla\cdot\mathbf{u} + w_z) = \rho D_p, \quad \text{and}$$

$$\theta_t + \mathbf{u}\cdot\nabla\theta + w\theta_z = D_\theta, \qquad (3.2)$$

where

$$\rho = \frac{p^{1/\gamma}}{\theta}. \qquad (3.3)$$

Here ρ, $\mathbf{u}$, w, p, and θ are the nondimensional density, horizontal velocity vector, vertical velocity, pressure, and potential temperature; $\mathbf{D}_u$, D_w, D_p, and D_θ represent unresolved scale closures for turbulent transport, radiation, etc.; $\mathbf{f}$ is the earth rotation unit vector; the "nabla-

TABLE 2. Dimensionless characteristic numbers based on the bulk microscales.

Number	Reference units
Mach	$M = \dfrac{v_{\mathrm{ref}}}{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}} \sim \dfrac{1}{64}$
Barotropic Froude	$\overline{\mathrm{Fr}} = \dfrac{v_{\mathrm{ref}}}{\sqrt{g\ell_{\mathrm{ref}}}} \sim \dfrac{1}{64}$
Bulk microscale Rossby	$\mathrm{Ro}_B = \dfrac{v_{\mathrm{ref}}}{\Omega\ell_{\mathrm{ref}}} \sim 5$

operator'' $\boldsymbol{\nabla} = (\partial_x, \partial_y, 0)$ indicates the horizontal gradient; and the subscripts $(\ )_\parallel$, $(\ )_\perp$ denote projections of a vector on the horizontal plane and on the vertical, respectively. The latter projections define $\mathbf{v}$ through $\mathbf{u} = \mathbf{v}_\parallel$ and $w = \mathbf{v}_\perp \cdot \mathbf{k}$, where $\mathbf{k}$ is the vertical unit vector. Table 2 defines the nondimensional characteristic numbers introduced and provides estimates of their typical orders of magnitude.

The reference length and time from Table 1 characterize the smallest scales considered in the present derivations. These are spatial scales comparable to the pressure-scale height h_{scale} and the associated advection timescale $t_{\mathrm{ref}} \sim \ell_{\mathrm{ref}}/v_{\mathrm{ref}}$. Notice, however, that longer timescales and a hierarchy of larger horizontal scales will be incorporated through the multiple-scale asymptotics approach. In fact, the synoptic-scale Rossby number mentioned in (3.1), will be introduced later on and will satisfy $\mathrm{Ro} = \varepsilon^2\mathrm{Ro}_B \ll 1$.

b. Distinguished limits

The governing equations in (3.2) involve a number of reasonably small or large nondimensional parameters. A multiparameter asymptotic analysis that treats each of these as independent leads to ambiguous results, because the limit for, say, two small parameters ε and δ generally depends on whether one first lets $\varepsilon \to 0$ and then $\delta \to 0$ or vice versa, or whether one considers a particular coupling between the two, called a ''distinguished limit.'' Here, this ambiguity is removed by introducing a single small parameter $\varepsilon \ll 1$ and the distinguished limits

$$M = \varepsilon^2 \hat{M}, \qquad \overline{\mathrm{Fr}} = \varepsilon^2\widehat{\mathrm{Fr}}, \qquad \mathrm{Ro}_B = \frac{1}{\varepsilon}\widehat{\mathrm{Ro}}, \tag{3.4}$$

with $\hat{M}$, $\widehat{\mathrm{Fr}}$, $\widehat{\mathrm{Ro}} = O(1)$ as $\varepsilon \to 0$.

This particular choice is compatible with the estimates in Table 1 for

$$\varepsilon \sim 1/8. \tag{3.5}$$

As in section 2, the derivations below apply for $\varepsilon \ll 1$ and the value in (3.5) is utilized for physical interpretation. For convenience of notation, all of the order $O(1)$ parameters $\hat{M}$, $\widehat{\mathrm{Ro}}$, $\widehat{\mathrm{Fr}}$, etc., are set to unity from here on. Notice, however, that these parameters are generally independent of each other and that their relative magnitudes can nontrivially influence actual solutions of the derived simplified asymptotic limit equations.

Below, it will be assumed that the nondimensional potential temperature deviates from a constant by order $O(\varepsilon^2)$ only. This scaling is supported by the estimates given earlier in section 2, and by the following nondimensionalization of the Brunt–Väisälä frequency N. A reference value typically found in the literature is $N \sim 2 \times 10^{-2}\ \mathrm{s}^{-1}$, which, nondimensionalized, yields

$$\left(\frac{\ell_{\mathrm{ref}}N}{v_{\mathrm{ref}}}\right)^2 \sim 100 \sim \varepsilon^{-2}. \tag{3.6}$$

Inserting the definition

$$N^2 = \frac{g}{\theta}\frac{\partial\theta}{\partial z} \sim \frac{g}{\theta_{\mathrm{ref}}}\frac{\delta\theta}{h_{\mathrm{scale}}}, \tag{3.7}$$

one finds, using $v_{\mathrm{ref}}/g\ell_{\mathrm{ref}} = \overline{\mathrm{Fr}} \sim \varepsilon^2$ and the convention that $\ell_{\mathrm{ref}} = h_{\mathrm{scale}}$,

$$\frac{\delta\theta}{\theta_{\mathrm{ref}}} \sim \varepsilon^2, \tag{3.8}$$

or, nondimensionally,

$$\theta = 1 + O(\varepsilon^2). \tag{3.9}$$

In (Klein 2000) the exact same scalings and distinguished limits from (3.4)–(3.9) are proposed as the basis for unified derivations of a large class of simplified model equations for midlatitude flows using multiple-scale asymptotic techniques. Besides showing that many midlatitude and near-equatorial flow regimes are properly described by the same asymptotic distinguished limits, in the present work the power of the systematic techniques is demonstrated by explicitly revealing nonlinear interactions across widely differing scales.

c. Expansion schemes for two distinct WTG regimes

Reduced models are derived here for scale interactions in two distinctly different tropical flow regimes using $\varepsilon \ll 1$ from (3.4) as a small expansion parameter. In the first regime, called the mesoscale/subplanetary regime (M/sP), the ε dependence of the solution $\mathbf{U}(t, \mathbf{x}, z; \varepsilon)$, where $\mathbf{U} = (p, \mathbf{u}, w, \theta)$, is expressed as

$$\mathbf{U}(t, \mathbf{x}, z; \varepsilon) = \sum_i \varepsilon^i \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^3 t, \varepsilon^2\mathbf{x}, \varepsilon^3\mathbf{x}, z). \tag{3.10}$$

The nondimensionalizations described above imply that the various arguments of the expansion functions $\mathbf{U}^{(i)}$ resolve roughly the length and timescales displayed in Table 3.

On the mesoscales, the effective Rossby number is of order unity so that rotational effects are not dominant in the horizontal momentum balance. These scalings correspond to the regime in section 2c for the equatorial shallow water equations.

The second regime is called synoptic/planetary (S/P) and it is characterized by a dominant equatorial geo-

TABLE 3. Length and timescales covered by the mesoscale/subplanetary regime.

Variable	Resolves (conservative estimates)	Description
$T_M = \varepsilon^2 t$	0.5–4 days	Mesoscale advection time
$\mathbf{X}_M = \varepsilon^2 \mathbf{x}$	200–1800 km	Mesoscale
$T_{\mathrm{Sea}} = \varepsilon^3 t$	4–34 days	Seasonal timescales
$\mathbf{X}_{\mathrm{sP}} = \varepsilon^3 \mathbf{x}$	1800–14 000 km	Subplanetary length scales

TABLE 4. Length and timescales for the synoptic/planetary regimes.

Variable	Resolves (conservative estimates)	Description
$T_{S,g} = \varepsilon^{3/2} t$	4–36 h	Synoptic gravity wave timescales
$T_S = \varepsilon^{5/2} t$	1.5–13 days	Synoptic advection timescales
$\mathbf{X}_S = \varepsilon^{5/2} \mathbf{x}$	600–5000 km	Synoptic scales
$\mathbf{X}_P = \varepsilon^{7/2} \mathbf{x}$	5000–40 000 km	Planetary length scales

strophic balance as in section 2b for the equatorial shallow water equations. The appropriate expansion in this regime reads

$$\mathbf{U}(t, \mathbf{x}, z; \varepsilon) = \sum_i \varepsilon^i \mathbf{V}^{(i)}(\varepsilon^{3/2} t, \varepsilon^{5/2} t, \varepsilon^{5/2}\mathbf{x}, \varepsilon^{7/2}\mathbf{x}, z). \tag{3.11}$$

Estimates of the corresponding resolved spatiotemporal scales are given in Table 4.

The distinguished limits introduced earlier for the Mach, Rossby, and Froude numbers then transform the nondimensional governing equations from (3.2), into

$$\begin{aligned}
&\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + w\mathbf{u}_z + (\varepsilon^4 y\mathbf{k}\times\mathbf{u} - \varepsilon w\mathbf{k}\times\mathbf{f}) \\
&\quad + \varepsilon^{-4}\frac{1}{\rho}\nabla p = \mathbf{S}_u, \\
&w_t + \mathbf{u}\cdot\nabla w + ww_z + \varepsilon(\mathbf{f}\times\mathbf{u}) \\
&\quad + \varepsilon^{-4}\frac{1}{\rho}p_z = S_w - \varepsilon^{-4}, \\
&p_t + \mathbf{u}\cdot\nabla p + wp_z + \gamma p(\nabla\cdot\mathbf{u} + w_z) = S_p, \quad \text{and} \\
&\theta_t + \mathbf{u}\cdot\nabla\theta + w\theta_z = S_\theta,
\end{aligned} \tag{3.12}$$

where the standard equatorial beta plane approximation for the variation of the Coriolis terms has been introduced, and y denotes the distance from the equator nondimensionalized by ℓ_{ref}. The characteristic length of variations of the Coriolis parameter is the earth radius a (a quarter of the earth's circumference), which is estimated here by

$$\frac{a}{h_{\mathrm{scale}}} \sim 600 \sim \varepsilon^{-3}. \tag{3.13}$$

As in section 2, the needed hierarchy of equations for the sequences of expansion functions $[\mathbf{U}^{(i)}]$, $[\mathbf{V}^{(i)}]$ from (3.10), (3.11) for $i = 0, 1, 2, \ldots$ is obtained by expanding the partial derivatives

$$\begin{aligned}
\text{M/sP} - \text{Regime:}\quad & \partial_t = \varepsilon^2\partial_{T_M} + \varepsilon^3\partial_{T_{\mathrm{Sea}}} \\
& \nabla = \varepsilon^2\nabla_M + \varepsilon^3\nabla_{\mathrm{sP}}, \\
\text{S/P} - \text{Regime:}\quad & \partial_t = \varepsilon^{3/2}\partial_{T_{S,g}} + \varepsilon^{5/2}\partial_{T_S} \\
& \nabla = \varepsilon^{5/2}\nabla_S + \varepsilon^{7/2}\nabla_P,
\end{aligned} \tag{3.14}$$

inserting them into the rescaled governing equations from (3.12) and collecting terms multiplied by like powers of ε. By averaging the resulting equations over the short-scale variables and applying the sublinear growth condition as explained in the previous section we extract the respective dependencies of the expansion functions on the extended sets of independent variables, $\{T_M, T_{\mathrm{Sea}}, \mathbf{X}_M, \mathbf{X}_{\mathrm{sP}}, z\}$ and $\{T_{S,g}, T_S, \mathbf{X}_S, \mathbf{X}_P, z\}$, respectively.

d. The meso- and seasonal/subplanetary scales

This section addresses flows on the 200–1800-km horizontal mesoscales and their interactions with motions on the subplanetary scale of 1800–14000 km. When restricted to the meso length and timescales alone, the analysis produces three-dimensional analogues (MEWTG^{3D}) of the weak temperature gradient shallow water theories (Held and Hoskins 1985; Browning et al. 2000; Sobel et al. 2001), and discussed in section 2 above. Considering the seasonal time and subplanetary length scales, which exceed the mesoscales by one order of magnitude in ε, the three-dimensional analog of the SPEWTG model from section 2 emerges.

1) A PRIORI ASSUMPTIONS AND ASYMPTOTIC EXPANSION SCHEME

The velocities, potential temperature, and pressure are expanded in ε according to

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)} + \varepsilon^2\mathbf{u}^{(2)} + \varepsilon^3\mathbf{u}^{(3)} + \cdots \\
w &= w^{(0)} + \varepsilon w^{(1)} + \varepsilon^2 w^{(2)} + \varepsilon^3 w^{(3)} + \cdots \\
p &= P_0(z) + \rho_0(z)[\varepsilon P_1(z) + \varepsilon^2 P_2(z) + \varepsilon^3\pi^{(3)} + \cdots] \\
\theta &= 1 + \varepsilon^2\Theta_2(z) + \varepsilon^3\theta^{(3)} + \cdots.
\end{aligned} \tag{3.15}$$

This ansatz implies a spatially homogeneous and, on the considered scales, time-independent background thermodynamic state characterized by a given potential temperature distribution $\overline{\theta}(z) = 1 + \varepsilon^2\Theta_2(z)$, and corresponding (hydrostatic) pressure and density distributions $P_0(z) + \varepsilon P_1(z) + \varepsilon^2 P_2(z)$ and $\rho_0(z) + \varepsilon\rho_1(z) + \varepsilon^2\rho_2(z)$, respectively. This is consistent with the earlier estimates in (3.9). From the viewpoint of climate dynamics, it is interesting that a detailed analysis shows that a general time dependence of $\theta^{(2)}$ can be treated, and that its temporal evolution depends on the presence of nonzero averages of energy source terms on planetary scales. For the sake of compactness of the presentation, these details will be described elsewhere. The particular choice of the pressure perturbation variables $p^{(i)} = \rho_0 \pi(i)$ has been borrowed from (Pedlosky 1987) as it simplifies some tedious details of the derivations.

Except for the above qualifications, the perturbation functions $\theta^{(i)}$, $\mathbf{u}^{(i)}$, $w^{(i)}$, $\pi^{(i)}$ depend on the multiple time and horizontal space coordinates from Table 3; that is,

$$(\theta^{(i)}, \mathbf{u}^{(i)}, w^{(i)}, \pi^{(i)}) = (\theta^{(i)}, \mathbf{u}^{(i)}, w^{(i)}, \pi^{(i)})(T_M, T_{\mathrm{Sea}}, \mathbf{X}_M, \mathbf{X}_{\mathrm{sP}}, z). \tag{3.16}$$

2) SELECTED STEPS OF THE DERIVATIONS

(i) Vertical momentum balance

The dominant effects of gravity on vertical momentum enforce hydrostatic balance up to $O(\varepsilon^4)$. These yield

$$p^{(0)} = P_0(z) = \left(1 - \frac{\gamma - 1}{\gamma} z\right)^{(\gamma-1)/\gamma} \tag{3.17}$$

for the leading-order pressure and

$$\pi^{(1)} \equiv 0, \qquad \frac{\partial \pi^{(i)}}{\partial z} = \theta^{(i)} \quad \text{for } (i = 2, 3),$$
$$\frac{\partial \pi^{(4)}}{\partial z} = \theta^{(4)} + \frac{\gamma - 1}{2}[P_2(z)]^2 \tag{3.18}$$

for the higher-order pressure perturbations. From the thermodynamic relations between pressure, density, and potential temperature it follows that

$$\rho_0(z) = P_0^{1/\gamma}(z), \qquad \rho^{(1)} \equiv 0 \quad \text{and} \quad \rho^2 \equiv \rho_2(z). \tag{3.19}$$

(ii) Mass balance

The immediate consequence of (3.19) and (3.12) is

$$(\rho_0 w^{(0)})_z = (\rho_0 w^{(1)})_z \equiv 0. \tag{3.20}$$

Suppose that $w^{(0)} \neq 0$ at some level $z = z_0$. Then, since $\rho_0(z) \to 0$ as $z \to \infty$, one would have $w^{(0)} = (w^{(0)}\rho_0)(z_0)/\rho_0(z) \to \infty$ as $z \to \infty$, which is clearly unphysical. Therefore, with analogous arguments for $w^{(1)}$,

$$w^{(0)} = w^{(1)} \equiv 0. \tag{3.21}$$

The next order yields the equation

$$\nabla_M \cdot \mathbf{u}^{(0)} + \frac{1}{\rho_0}(\rho_0 w^{(2)})_z = 0. \tag{3.22}$$

(iii) Horizontal momentum balance

The dominant factor of ε^{-4} multiplying the pressure gradient in (3.12) and, due to (3.21), the absence of any corresponding balancing term up to order $O(1)$ imply that horizontal pressure gradients must be systematically weak. In fact, by applying appropriate sublinear growth conditions it follows that

TABLE 5. Length and timescales for the subsynoptic and subplanetary WTG regimes.

Variables		Timescale	Horizontal scale	Regime
(I)	T_M, $\mathbf{X}_M$, z	0.5–4 days	200–1800 km	MEWTG
(II)	T_{Sea}, $\mathbf{X}_{\mathrm{sP}}$, z	4–34 days	1800–14 000 km	SPEWTG

$$\nabla_M \pi^{(i)} \equiv 0 \qquad (i = 0, 1, 2, 3),$$
$$\nabla_{\mathrm{sP}} \pi^{(i)} \equiv 0 \qquad (i = 0, 1, 2). \tag{3.23}$$

At higher orders, the prognostic momentum equations are given by

$$\partial_{T_M}\mathbf{u}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla_M \mathbf{u}^{(0)} + w^{(2)}\partial_z \mathbf{u}^{(0)} + \beta Y_M \mathbf{k} \times \mathbf{u}^{(0)} + \nabla_M \pi^{(4)} = S_u^{(2)} - \nabla_{\mathrm{sP}} \pi^{(3)}, \tag{3.24}$$

and

$$\partial_{T_M}\mathbf{u}^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_M \mathbf{u}^{(1)} + w^{(2)}\partial_z \mathbf{u}^{(1)} + \mathbf{u}^{(1)} \cdot \nabla_M \mathbf{u}^{(0)} + \partial_{T_{\mathrm{Sea}}}\mathbf{u}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla_{\mathrm{sP}} \mathbf{u}^{(0)} + w^{(3)}\partial_z \mathbf{u}^{(0)} + \beta Y_M \mathbf{k} \times \mathbf{u}^{(1)} + \nabla_M \pi^{(5)} = S_u^{(3)} - \nabla_{\mathrm{sP}} \pi^{(4)}. \tag{3.25}$$

(iv) Potential temperature transport

Expanding the potential temperature transport equation, the first nontrivial result is the basic WTG balance

$$w^{(2)} \frac{d}{dz}\Theta_2 = S_\theta^{(4)}. \tag{3.26}$$

It states that mass elements move quasiinstantaneously toward their new vertical level of neutral buoyancy under local heat addition. The third-order potential temperature transport is determined by

$$\partial_{T_M}\theta^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_M \theta^{(3)} + w^{(2)}\partial_z \theta^{(3)} + w^{(3)} \frac{d}{dz}\Theta_2 = S_\theta^{(5)}. \tag{3.27}$$

3) KEY RESULTS AND DISCUSSION

(i) Specializations for selected scales

Here, specializations of (3.17)–(3.42) to only one time and one horizontal scale each are discussed. Such specializations correspond directly with classical scaling analysis, where the dominant time and horizontal spatial scales of solutions are presumed. In this paragraph, we will concentrate on the combinations of independent variables and associated physical regimes displayed in Table 5.

(ii) The "classical" mesoscale equatorial WTG: MEWTG^{3D}

The first specialization in Table 5 using the replacements

$$[\mathbf{u}^{(0)}, w^{(2)}, \pi^{(4)}] \to (\mathbf{u}, w, \pi),$$

$$(T_M, \mathbf{X}_M) \to (t, \mathbf{x}), \quad \text{and}$$
$$\boldsymbol{\nabla}_M \to \boldsymbol{\nabla}, \tag{3.28}$$

leads to a system of "weak temperature gradient" equations analogous to those derived, for example, in Browning et al. (2000) and Sobel et al. (2001):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} + w\partial_z \mathbf{u} + \boldsymbol{\nabla}\pi + \beta y \mathbf{k} \times \mathbf{u} = \mathbf{S}_u,$$
$$\boldsymbol{\nabla} \cdot \rho_0 \mathbf{u} + \partial_z(\rho_0 w) = 0, \quad \text{and}$$
$$w\frac{d\Theta_2}{dz} = S_\theta. \tag{3.29}$$

They describe flows that are balanced with respect to much faster gravity waves and that are driven by the buoyancy-induced vertical motions resulting from energy addition into a dry stable atmosphere.

(iii) Seasonal subplanetary equatorial WTG: SPEWTG^{3D}

Here, as in section 2c, an anisotropic scaling w.r.t. the zonal and meridional directions is required in order to restrict to the near-equatorial belt. On these large scales the relevant averaged heat source terms must be one order of magnitude smaller than on the smaller "classical WTG" scales, which is consistent with the discussions in section 2. With the replacements

$$(u^{(0)}, v^{(1)}, w^{(3)}, \pi^{(4)}) \to (u, v, w, \pi), \quad \text{and}$$
$$(T_{\text{Sea}}, X_{\text{sP}}, Y_M) \to (t, x, y), \tag{3.30}$$

one obtains the three-dimensional extension of the SPEWTG from section 2; namely,

$$\partial_t u + v\partial_y u + w\partial_z u - \beta y v + \partial_x \pi = S_u,$$
$$\beta y u + \partial_y \pi = S_v,$$
$$\partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0, \quad \text{and}$$
$$w\frac{d\Theta_2}{dz} = S_\Theta. \tag{3.31}$$

As pointed out in the shallow water context in section 2, these equations have one particularly interesting feature related to midlatitude connections. As shown through explicit calculations below, linearization of these equations for *arbitrary* vertical structure reveals a linear dispersion relation that exactly matches that of *long wavelength barotropic midlatitude Rossby waves.* Thus, the equations in (3.31) describe quite general equatorial flow structures, which can resonate naturally with zonally large scale midlatitude synoptic-scale flows.

4) Linear dispersion relations for midlatitude connections

For the present discussion it suffices to consider the linearized source-free version of the MEWTG model equations in (3.29):

$$\partial_t u - \beta y v + \partial_x \pi = 0,$$
$$\partial_t v + \beta y u + \partial_y \pi = 0,$$
$$\partial_x u + \partial_y v = 0, \quad \text{and}$$
$$w = 0. \tag{3.32}$$

These equations admit the introduction of a streamfunction with arbitrary vertical structure

$$\Psi(t, x, y, z) = A(z)\psi(t, x, y), \tag{3.33}$$

so that the divergence constraint from (3.32) is trivially satisfied through

$$u = -A(z)\partial_y \psi, \quad \text{and} \quad v = A(z)\partial_x \psi. \tag{3.34}$$

Cross differentiation of the momentum equations in (3.32) then yields

$$\partial_t(\Delta\psi) + \beta\partial_x\psi = 0, \tag{3.35}$$

which is the linearized streamfunction equation for midlatitude *barotropic* Rossby waves. The same ansatz with arbitrary vertical structure in (3.33) applied to the linearized free wave solutions of the SPEWTG equations in (3.31) yields the equation

$$\partial_t \psi_{yy} + \beta\psi_x = 0. \tag{3.36}$$

As discussed below (2.26), this equation has the same dispersion relation as the one for dynamics of long wavelength barotropic Rossby waves.

e. Synoptic/planetary dynamics

This section addresses flows on the 600–5000-km horizontal "synoptic" scales and motions on the "planetary" scales of 5000–40 000 km. These larger scales support planetary equatorial waves, such as the Kelvin and long wavelength equatorial Rossby waves. Here, it is shown in the present setting of the full compressible primitive 3D flow equations, how to develop reduced equations that describe the generation of these wave modes through cumulative synoptic-scale effects.

1) Expansion scheme

In this regime, the potential temperature, velocity, and pressure are expanded as

$$\mathbf{u} = \mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)} + \varepsilon^2\mathbf{u}^{(2)} + \varepsilon^3\mathbf{u}^{(3)} + \cdots$$
$$w = \varepsilon^{5/2}w^{(5/2)} + \varepsilon^{7/2}w^{(7/2)} + \cdots$$
$$p = P_0(z) + \rho_0(z)[\varepsilon^2 P_2(z) + \varepsilon^3\pi^{(3)} + \cdots]$$
$$\theta = 1 + \varepsilon^2\Theta_2(z) + \varepsilon^3\theta^{(3)} + \cdots. \tag{3.37}$$

The arguments leading to elimination of $\theta^{(1)}$, $\pi^{(1)}$, $w^{(3/2)}$, etc., are analogous to those given in the last section and will not be repeated here.

The perturbation functions $\theta^{(i)}$, $\mathbf{u}^{(i)}$, $w^{(i)}$, $\pi^{(i)}$ depend on the coordinates from Table 4; that is,

$$(\theta^{(i)}, \mathbf{u}^{(i)}, w^{(i)}, \pi^{(i)}) = (\theta^{(i)}, \mathbf{u}^{(i)}, w^{(i)}, \pi^{(i)})(T_{S,g}, T_S, \mathbf{X}_S, X_P, z). \quad (3.38)$$

Notice that, as in section 2, only the zonal component of $\mathbf{X}_P = (X_P, Y_P)$ is included here. The variables in $\mathbf{X}_P$ resolve length scales of 5000 ~ 40 000 km, so that inclusion of the meridional component Y_P would force us to include midlatitude effects in the analysis. As in section 2b this particular choice is justified by the results to be described below and the asymptotic regimes obtained are in fact compatible with the setup from (3.38). The issue of a consistent matching of equatorial dynamics to the midlatitudes remains, however, and the authors plan to address it elsewhere in the future.

2) Selected steps of the derivations

(i) Vertical momentum balance

The arguments and derivations given in (3.17)–(3.19) carry over identically to the present regime.

(ii) Horizontal momentum balance

The first two nontrivial horizontal momentum equations read

$$\partial_{T_{S,g}}\mathbf{u}^{(0)} + \beta Y_S \mathbf{k} \times \mathbf{u}^{(0)} + \nabla_S \pi^{(3)} = S_u^{3/2}, \quad \text{and} \quad (3.39)$$

$$\partial_{T_{S,g}}\mathbf{u}^{(1)} + \partial_{T_S}\mathbf{u}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla_S \mathbf{u}^{(0)} + w^{(5/2)}\partial_z \mathbf{u}^{(0)} + \beta Y_S \mathbf{k} \times \mathbf{u}^{(1)} + \nabla_S \pi^{(4)} + \mathbf{i}\partial_{X_P}\pi^{(3)} = S_u^{5/2}. \quad (3.40)$$

Here, $\mathbf{i}$ is the unit vector pointing in the zonal direction.

(iii) Potential temperature transport

The inclusion of a fast timescale via the coordinate $T_{s,g} = \varepsilon^{3/2} t_{\text{ref}}$ in the expansion leads to a fast time prognostic equation for $\theta^{(3)}$, which is in contrast to the synoptic-scale regime discussed earlier:

$$\partial_{T_{S,g}}\theta^{(3)} + w^{(5/2)}\frac{d}{dz}\Theta_2 = S_\theta^{(9/2)}. \quad (3.41)$$

The time derivative in this equation will give rise to internal gravity wave modes in the asymptotic solutions for this regime. The next order yields the potential temperature perturbation equations:

$$\partial_{T_{S,g}}\theta^{(4)} + \partial_{T_S}\theta^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_S \theta^{(3)} + w^{(5/2)}\partial_z \theta^{(3)} + w^{(7/2)}\frac{d}{dz}\Theta_2 = S_\theta^{(11/2)}. \quad (3.42)$$

(iv) Mass balance

To leading order and first order, mass conservation yields the divergence constraints

$$\nabla_S \cdot \mathbf{u}^{(0)} + \frac{1}{\rho_0}\partial_z(\rho_0 w^{(5/2)}) = 0, \quad \text{and} \quad (3.43)$$

$$\partial_{X_P} u^{(0)} + \nabla_S \cdot \mathbf{u}^{(1)} + \frac{1}{\rho_0}\partial_z(\rho_0 w^{(7/2)}) = 0. \quad (3.44)$$

3) Key results and discussion

In this section two important multiple-scale regimes are discussed separately. The first considers synoptic length scales only ($\partial_{X_P} \equiv 0$), while including fast gravity waves as well as the slower advective timescales. As in section 2a, this will lead to the classical linearized equatorial wave equations at leading order on the fast timescale. Second, the approach yields the three-dimensional version of the intraseasonal planetary equatorial dynamics and the simpler synoptic equatorial WTG approximation, which are derived in section 2b for the equatorial shallow water models. To this end synoptic timescales only ($\partial_{T_{S,g}} \equiv 0$) are considered, but both the syn optic and the planetary length scales are included.

The derivations may conveniently be summarized by introducing the replacements

$$[\theta^{(3)}, \mathbf{u}^{(0)}, w^{(5/2)}, w^{(7/2)}, \pi^{(3)}, \pi^{(4)}] \to (\theta, \mathbf{u}, \mathbf{u}', w, w', \pi, \pi'),$$
$$[S_\theta^{(9/2)}, S_\theta^{(11/2)}, \mathbf{S}_u^{(3/2)}, \mathbf{S}_u^{(5/2)}] \to (S_\theta, S'_\theta, \mathbf{S}_u, \mathbf{S}'_u),$$
$$(T_{S,g}, T_S) \to (\tau, t),$$
$$(\mathbf{X}_S, X_P) \to (x, y, X), \quad \text{and}$$
$$\nabla_S \to \nabla. \quad (3.45)$$

(i) Linear equatorial wave equations (LEWE^{3D})

Here, solutions are considered on fast gravity wave timescales and synoptic spatial scales. At leading order the three-dimensional generalization of the classical linear equatorial wave equations mentioned in (2.17) from section 2 emerge immediately in a similar fashion:

$$\partial_\tau \theta + w\frac{d\Theta_2}{dz} = S_\theta,$$
$$\partial_\tau u - \beta y v + \partial_x \pi = S_u,$$
$$\partial_\tau v + \beta y u + \partial_y \pi = S_v,$$
$$\partial_x u + \partial_y v + \frac{1}{\rho_0}\partial_z(\rho_0 w) = 0, \quad \text{and}$$
$$\partial_z \pi = \theta \quad (3.46)$$

The homogeneous solutions of (3.46) describe standard linear equatorial wave dynamics for the hydrostatic primitive equations. The underlying geostrophically balanced quasi-steady limit dynamics for (3.46) plays a central role in the next sections.

(ii) Intraseasonal planetary equatorial dynamics (IPESD^{3D}, SEWTG^{3D}, and QLELWE^{3D})

Here the synoptic equatorial WTG approximation (SEWTG^{3D}) is shown to interact with quasi-linear equatorial longwave equations (QLELWE^{3D}) across multiple spatial scales to produce intraseasonal planetary effects.

With the replacements from (3.45), and after dropping the fast time derivatives $\partial_{T_{s,g}}$, the leading-order equation system from (3.17)–(3.19) and (3.39) to (3.43) reads

$$\partial_z \pi = \theta,$$

$$w\frac{d\Theta_2}{dz} = S_\theta,$$

$$-\beta y v + \partial_x \pi = S_u,$$

$$\beta y u + \partial_y \pi = S_v, \quad \text{and}$$

$$\partial_x u + \partial_y v + \frac{1}{\rho_0}\partial_z(\rho_0 w) = 0. \tag{3.47}$$

These equations describe hydrostatic balance, the generation of vertical motions by heat sources forcing particles to move toward their individual levels of neutral buoyancy, horizontal geostrophic balance, and horizontal flow divergences induced by the thermally driven vertical motions as a consequence of mass conservation.

It is instructive to consider the source terms in (3.47) as given externally for the time being, in which case the equations are linear in u, v, w, π, θ. General solutions then consist of superpositions of particular and homogeneous solutions.

One particular solution u^p, v^p, w^p, π^p, θ^p is given through the following relations. First, the potential temperature transport equation in (3.47) and cross differentiation of the horizontal momentum balance equations yield explicit expressions for the vertical and meridional velocity components:

$$w^p = \frac{S_\theta}{d\Theta/dz},$$

$$v^p = \frac{1}{\beta}(S_{v,x} - S_{u,y}) + \frac{y}{\rho_0}\left(\frac{\rho_0 S_\theta}{d\Theta/dz}\right)_z. \tag{3.48}$$

The divergence constraint in (3.47) then becomes an ordinary differential equation in x:

$$\partial_x u^p = -\partial_y v^p - \frac{1}{\rho_0}\partial_z(\rho_0 w^p), \tag{3.49}$$

for the zonal velocity. Letting $\overline{(\cdot)}$ denote averaging with respect to the fast variable x, a unique solution for u^p is specified by requiring

$$\beta y \overline{u^p} = \overline{S_v}. \tag{3.50}$$

Given these explicit representations of the velocity field for the particular solution, the horizontal momentum balance determines the pressure field π^p except for an arbitrary function of (t, z) via

$$\partial_x \pi^p = S_u - \beta y v^p, \qquad \partial_y \pi^p = S_v + \beta y u^p. \tag{3.51}$$

Here a unique solution is selected by setting

$$\overline{\pi^p}(t, 0, z) \equiv 0, \tag{3.52}$$

and finally,

$$\theta^p = \partial_z \pi^p. \tag{3.53}$$

Note that, in analogy with the shallow water case, there is a constraint on the spatial structure of the source terms. It is obtained by averaging the horizontal momentum balance from (3.51) with respect to x:

$$\overline{S_u} + y\left[-\partial_y \overline{S_v} + \frac{\beta y}{\rho_0}\left(\frac{\rho_0 \overline{S_\theta}}{d\Theta/dz}\right)_z\right] = 0. \tag{3.54}$$

From here on u^p, v^p, w^p, π^p, and θ^p denote the particular solutions (3.48) to (3.53), so that they are known functionals of the source terms S_u, S_v, and S_θ. The latter are assumed to satisfy (3.54).

The reader may verify that homogeneous solutions to (3.47) have the following structure:

$$v = w \equiv 0, \qquad (\theta, u, \pi) = (\Theta, U, P)(t, y, z), \tag{3.55}$$

where Θ, U, P are arbitrary functions except for the hydrostatic and meridional geostrophic constraints

$$\partial_y P = -\beta y U, \qquad \partial_z P = \Theta. \tag{3.56}$$

As usual in geostrophically balanced systems, the leading order set of equations determines the spatially balanced structure of solutions, whereas their temporal evolution must be derived through sublinear growth conditions for the next-order perturbations. The necessary calculations closely follow the procedures outlined in section 2b, so that the details are omitted here. From (3.40), (3.41), and (3.44) and hydrostatic balance, it follows that

$$\overline{D_t^p}U + \partial_X P - \beta y \overline{v'} = \overline{S_u'} - \overline{D_t^p u^p},$$

$$\overline{D_t^p}\Theta + \overline{w'}\frac{d\Theta_2}{dz} = \overline{S_\theta'} - \overline{D_t^p \theta^p},$$

$$\beta y U + \partial_y P = 0,$$

$$\partial_z P = \Theta, \quad \text{and}$$

$$\partial_X U + \partial_y \overline{v'} + \frac{1}{\rho_0}\partial_z(\rho_0 \overline{w'}) = 0, \tag{3.57}$$

where

$$D_t^p = \partial_t + u^p\partial_x + v^p\partial_y + w^p\partial_z, \quad \text{and}$$

$$\overline{D_t^p} = \partial_t + \overline{v^p}\partial_y + \overline{w^p}\partial_z. \tag{3.58}$$

This completes the summary of the derivation of the three-dimensional version of the intraseasonal planetary equatorial synoptic-scale dynamics equations. It is worth emphasizing here again, following the discussion from section 2, that the source terms in the first two equations include the nonlinear averaged effects of synoptic-scale advection, and that these terms can be evaluated explicitly given the particular solutions to the

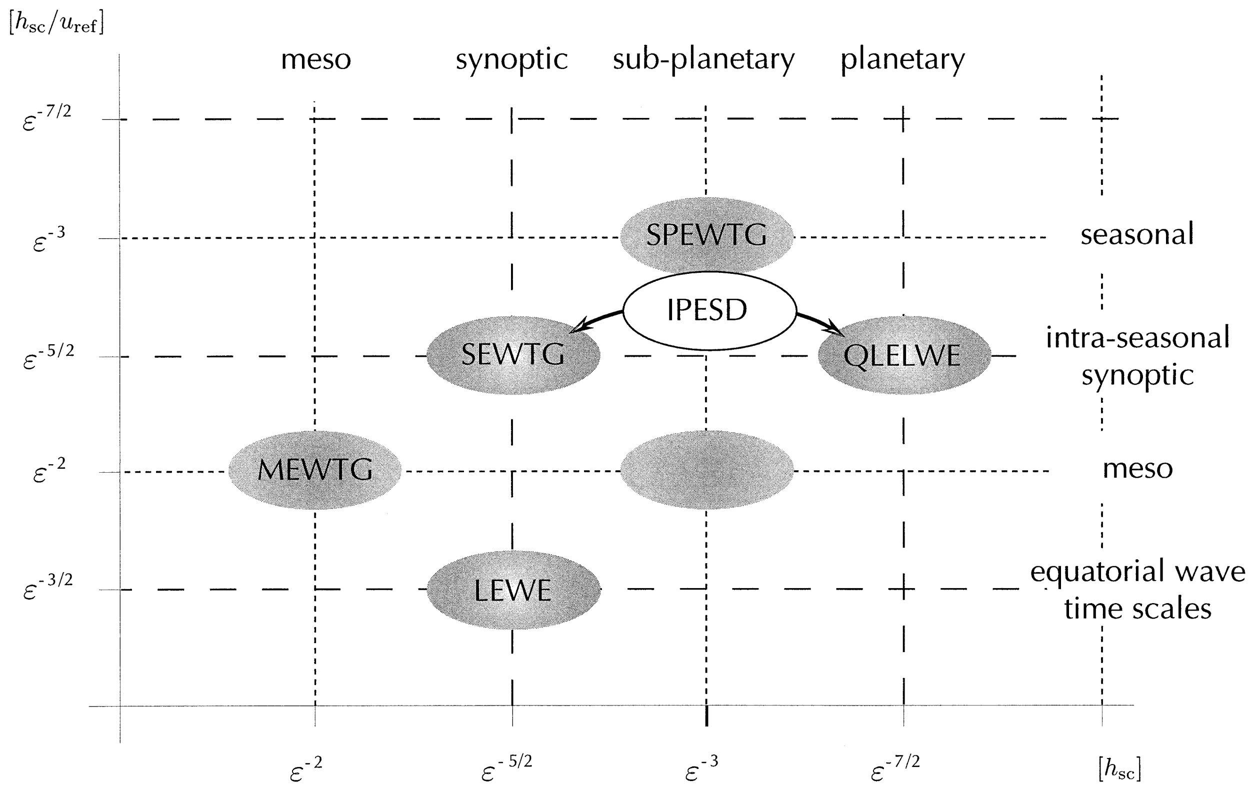

FIG. 1. Scale map of the equatorial flow regimes discussed in this paper.

leading-order system as derived above. Thus, the equations here are the full three-dimensional version of the new IPESD theory, which include the generation of planetary equatorial waves by the accumulation of nonlinear transport on synoptic scales. The equations in (3.57) differs significantly from the IPESD equations in section 2 through the explicit vertical advection in (3.58).

In analogy with the shallow water case discussed in section 2, one obtains the synoptic planetary equatorial weak temperature gradient approximation by neglecting the planetary-scale dependencies, that is, by dropping $\partial_X P$ and $\partial_X U$ in the zonal momentum and mass balance equations in $(3.57)_1$ and $(3.57)_5$, respectively. With these simplifications, these equations become the three-dimensional generalization of steady Gill models including new explicit prescriptions for the temporal evolution of the (quasi-)steady solutions.

Another specialization yields the three-dimensional version of the quasi-linear equatorial longwave equations. To this end, consider weak source terms, so that $S_u = S_v = S_\theta \equiv 0$, and as a consequence, the particular solution to the quasi-steady leading-order problem becomes trivial: $(\underline{u^p}, v^p, w^p, \theta^p, \pi^p) \equiv 0$. In this case the operators D_t^p, $\overline{D_t^p}$ reduce to the partial time derivative ∂_t, and the only source $\underline{\text{terms}}$ left in (3.57) are the perturbation source terms $\overline{S'_u}$, $\overline{S'_\theta}$. The equations then describe the generation of linear equatorial long waves by weak momentum and energy sources.

4. Concluding discussion

Several simplified models for the interaction across multiple space and/or timescales in the Tropics have been derived here through systematic multiscale perturbation theory from applied mathematics. Figure 1 summarizes these results by indicating for each of the derived simplified models the asymptotic scalings of the nondimensional space and time coordinates in terms of the single small expansion parameter ε. The latter has been identified through judiciously chosen distinguished limits for the Rossby, Froude, and Mach numbers in section 3 for the primitive equations. This same-scale map applies to the equatorial shallow water models discussed in section 2, although this fact was not emphasized there for pedagogical simplicity. Furthermore, additional model dynamics for the interaction across multiple space and/or timescales can be developed from the systematic procedures as needed.

The new IPESD models describe multiscale interactions among the synoptic/planetary scales, indicated by the coarsely dashed subgrid in the graph. These interactions involve the self-consistent quasi-linear coupling of generalized synoptic-scale steady Gill models and equatorial long waves; thus, these models have both WTG heating effects and simultaneously transient long-wave equatorial Kelvin and Rossby waves in their dynamics. Browning et al. (2000) have reasoned through the equatorial mesoscale WTG equations, derived here briefly in section 3 as the (MEWTG^{3D}) model, that the observational role of equatorial Kelvin and Rossby waves are exaggerated by using these modes in basis expansions (Heckley and Gill 1984). The derivation of the IPESD models from the compressible primitive equations in section 3 above indicates that their reasoning is not correct; in analogy with the single time-scale–multiple space-scale theory for weakly compressible flows from Klein (1995), these IPESD models reveal how the larger equatorial synoptic scales couple with the equatorial planetary scales self-consistently on the same timescale, and generate equatorially trapped planetary waves on intraseasonal timescales. The IPESD models have genuine potential as simplified diagnostic and prognostic models for the tropical intraseasonal oscillation when coupled with nonlinear feedback from radiation, and the boundary layer through moisture processes and nonlinear aerodynamic drag. The explicit treatment of moisture leads to stiff source terms, which are analogous to those occurring in other disciplines such as turbulent combustion, where techniques have been developed to treat such behavior (Majda and Souganidis 1994; Bourlioux and Majda 2000). All of these aspects, as well as simple illustrative solutions of the IPESD models, are currently being developed by the authors and will be published in the near future.

The finely dashed subgrid in the Fig. 1 indicates the meso-/subplanetary range of scales. The treatment here recovers the ''classical'' mesoscale equatorial WTG approximation from Sobel et al. (2001) in the shallow water context, and its three-dimensional generalization has been given in section 3. The mesoscales are characterized by the Charney inertial scale. Fast gravity waves are only weakly affected in this regime by Coriolis effects, and this hierarchy of scales in fact does not support equatorial Kelvin and Rossby waves. On the larger seasonal/subplanetary scales, we have discovered an additional ''semigeostrophic'' WTG regime, called here the seasonal/subplanetary equatorial WTG. It was established here that the MEWTG and SPEWTG model equations can have arbitrary vertical structure yet have solutions with the linear dispersion relation of barotropic Rossby waves. This feature makes these models attractive ingredients for theories of tropospheric midlatitude connections with the Tropics. The shaded region in Fig. 1 denotes a large-scale regime with potentially important multiple space scale interaction with MEWTG. The authors are currently pursuing these developments and will report on them elsewhere in the near future. Note in Fig. 1, that the IPESD models are separated from either the MEWTG or the SPEWTG models by roughly a factor of three in spatial behavior but simultaneously also a factor of three in temporal behavior. Further study is needed to decide whether these balanced models can discriminate between different physical effects in observations and simulations on these different scales, but this does occur in other physical problems with modest scale separation (Klein and Knio 1995; Bourlioux and Majda 2000).

The general systematic approach developed here is also useful for developing numerical procedures that accurately reflect the multiple scalings of the physics. One of the authors (Klein 2000) is currently developing such an approach for the midlatitudes on mesoscales, following earlier successful developments in other contexts (Klein 1995; Klein and Knio 1995; Worlikar et al. 1998; Schneider et al. 1999; Roller et al. 2002). Clearly, this is also an interesting research direction for the Tropics, especially given the poor capability of contemporary GCM's to resolve convectively coupled tropical waves.

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