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LOW FROUDE NUMBER LIMITING DYNAMICS FOR STABLY STRATIFIED FLOW WITH SMALL OR FINITE ROSSBY NUMBERS

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Abstract

Recent numerical simulations reveal remarkably different behavior in rotating stably stratified fluids at low Froude numbers for finite Rossby numbers as compared with the behavior at both low Froude and Rossby numbers. Here the reduced low Froude number limiting dynamics in both of these situations is developed with complete mathematical rigor by applying the theory for fast wave averaging for geophysical flows developed recently by the authors. The reduced dynamical equations include all resonant triad interactions for the slow (vortical) modes, the effect of the slow (vortical) modes on the fast (inertial gravity) modes, and also the general resonant triad interactions among the fast (internal gravity) waves. The nature of the reduced dynamics in these two situations is compared and contrasted here. For example, the reduced slow dynamics for the vortical modes in the low Froude number limit at finite Rossby numbers includes vertically sheared horizontal motion while the reduced slow dynamics in the low Froude number and low Rossby number limit yields the familiar quasigeostrophic equations where such vertically sheared motion is completely absent - in fact, such vertically sheared motions participate only in the fast dynamics in this quasigeostrophic limit. The use of Ertel's theorem on conservation of potential vorticity is utilized, for example, in studying the limiting behavior of the rotating Boussinesq equations with general slanted rotation and unbalanced initial data. Other interesting physical effects such as those of varying Prandtl number on the limting dynamics are also developed and compared here.

Key Words: strongly stratified flow, vortical modes, gravity waves, resonances.

1. INTRODUCTION

Rotating strongly stratified flows are ubiquitous features of the dynamics of the atmosphere and the ocean. The degree of stratification is measured by the Froude number, Fr, which characterizes the ratio of the buoyancy time to the velocity advective time scale while the degree of rotation is measured by the Rossby number, Ro, which characterizes the ratio of the rotation time to the advective time scale.

The universal features of the velocity spectrum on mesoscales in the lower atmosphere (Gage, 1979; Gage and Nastrom, 1986) as well as the universal wave spectrum on somewhat smaller scales in the ocean (Garrett and Munk, 1979) have inspired a large theoretical effort attempting to explain these phenomena. Analytical efforts have focused on the three-wave resonant nonlinear interactions among internal gravity waves (McComas and Bretherton, 1977; Muller, et. al., 1986) and more recently on three wave interactions for an individual triad with one gravity waves and a vortical mode (Lelong, 1989; Lelong and Riley, 1991). Following the pioneering work of Riley et. al. (1981), recent large numerical simulations in idealized periodic geometry have studied solutions of the rotating Boussinesq equations with either continual forcing (Herring and Metais, 1989; Ramsden and Holloway, 1992; Metais et. al. 1994) or free decay (Metais and Herring, 1989; Bartello, 1995) at a variety of Froude and Rossby numbers in an attempt to explain the universal observed spectra in an idealized setting. These numerical experiments reveal remarkable differences in the behavior of strongly stratified flows with low Froude number and finite Rossby number as compared with such flows at both low Froude number and low Rossby number.

The purpose of this paper is to develop in detail the reduced limiting dynamics dynamics for solutions of the rotating Boussinesq equations both at low Froude number and finite Rossby number as well as for the low Froude number and low Rossby number limit and then to compare and contrast these dynamics to provide a general analytical framework for explaining some of the remarkable features observed in the numerical simulations. To achieve this, we apply the recent mathematically rigorous theory for averaging over fast waves in geophysical flows developed by the authors (Embid and Majda, 1996). This theory involves a modification to account for the strong dispersion in geophysical flows of techniques developed earlier in the mathematical study of incompressible limits for compressible flows (Klainerman and Majda, 1981; Majda, 1984; Schochet, 1994) as well as judicious use of Ertel's theorem on conservation of potential vorticity. The general theory yields reduced limiting dynamics which include resonant triad interactions for the slow (vortical) modes, the effect of the slow (vortical) modes on the fast internal (gravity) modes and also the general resonant triad interactions among internal gravity waves from earlier work (McComas and Bretherton, 1977).

We briefly summarize some crucial differences in the limiting dynamics. The reduced slow dynamics for the vortical modes in the low Froude number limit at finite Rossby numbers includes vertically sheared horizontal motion while the reduced slow dynamics in the low Froude number and low Rossby number limit yields the familiar quasigeostrophic equations where such vertically sheared horizontal motion is completely absent – in fact, such vertically sheared motions participate only in the fast dynamics through resonant interactions in this quasigeostrophic limit. The reduced dynamics also explains the conservation in time of the energy ratio between vortical modes and gravity modes observed in decaying numerical simulations at low Froude numbers (Metais and Herring, 1989). Furthermore, the reduced equations derived here for the slow-fast vortical and gravity wave interactions include those derived earlier by Lelong (1989), Lelong and Riley (1991), and Bartello (1995) as special cases.

Next we outline the contents of the remainder of this paper. Section 2 is a preliminary section where we discuss the nondimensional equations for stably stratified flow with slanted rotation, Ertel's theorem for potential vorticity, and the framework for fast averaging for geophysical flows (Embid and Majda, 1996). In section 3 we develop the low Froude number limiting dynamics at finite Rossby number while the low Froude number and low Rossby number limiting dynamics are developed in section 4. In section 5, we compare and contrast the form of the limiting dynamics in these two situations and end the paper in section 6 with some concluding discussion. We also emphasize differences in the limiting dynamics with varying Prandtl numbers. This is interesting for comparing laboratory experiments in rotating stratified flow (Fincham, et al., 1996) where typically, the Prandtl number has order 200, with numerical simulations where the Prandtl number is one and also to understand anisotropic vector eddy diffusivity effects in geophysical flows.

The work presented here is also interesting from the viewpoint of studying the effect of general unbalanced initial data with large amplitude gravity waves on the slow dynamics described by quasigeostrophic flow. The theoretical results presented here establish rigorously that in a suitable averaged sense involving space-time filtering through the weak topology, the underlying dynamics always observed is given by the quasigeostrophic dynamics even without dissipation. These facts are emphasized in the earlier paper by the authors (Embid and Majda, 1996) which also includes a detailed study of the rapidly rotating shallow water equations. A sketch of this argument, which utilizes Ertel's theorem, is presented in section 4 for the general situation with slanted rotation.

2. PRELIMINARIES

2.1 Rotating Stratified Flow with Slanted Rotation

To study the dynamics of a rotating stratified flow we utilize the Boussinesq approximation for a stably stratified rotating fluid

$$\frac{D\vec{v}}{Dt} + f\vec{\eta} \times \vec{v} + \rho_b^{-1}\rho\nabla\phi + \rho_b^{-1}\rho\vec{e}_3 = \mu\rho_b^{-1}\Delta\vec{v}$$

$$\frac{D\rho}{Dt} - bw = D\Delta\rho \tag{2.1}$$

$$div\ \vec{v} = 0$$

where $\vec{v}$ is the velocity, ρ is the density fluctuation, and ϕ is the pressure. It is convenient to decompose the velocity field $\vec{v}$ in terms of its horizontal component $\vec{v}_H$ and its vertical component w, $\vec{v} = (\vec{v}_H, w) = (v_1, v_2, w)$. The density $\tilde{\rho}$ is decomposed in terms of a reference density $\bar{\rho}$ and the density fluctuation ρ

$$\tilde{\rho} = \bar{\rho} + \rho$$

$$\bar{\rho} = \rho_b - bx_3. \tag{2.2}$$

For simplicity we assume that the reference density profile $\bar{\rho}$ decreases linearly with height , so that $b > 0$ for stable stratification. The Coriolis term is $2\vec{\Omega} = f\vec{\eta}$, where the direction of the axis of rotation is given by the unit vector $\vec{\eta}$. For the general case of a rotating fluid with slanted rotation, the axis of rotation $\vec{\eta}$ does not coincide with the vertical direction of gravity $\vec{e}_3$, $\vec{\eta} \neq \vec{e}_3$. On the other hand, the standard case of vertical rotation corresponds to $\vec{\eta} = \vec{e}_3$. Finally, μ is the viscosity and D is the mass diffusion coefficient.

In order to non-dimensionalize the rotating Boussinesq equations with slanted rotation in (2.1) we introduce characteristic scales for the physical variables; L is the length scale in both the horizontal and the vertical directions, U is the velocity scale, ρ_b is the scale for the reference density, $\rho_b B$ is the scale for the density fluctuation, and p is the scale for the pressure. In addition, the buoyancy (Brunt-Väisälä) frequency N is given by

$$N = \left(-\frac{g}{\rho_b}\frac{\partial\bar{\rho}}{\partial x_3}\right)^{1/2} = \left(\frac{gb}{\rho_b}\right)^{1/2} .$$

With this choice of scales the non-dimensional form of the rotating Boussinesq equations is

$$\frac{\partial\vec{v}}{\partial t} + (Ro)^{-1}\vec{\eta} \times \vec{v} + \overline{P}\ \nabla\phi + \Gamma\rho\vec{e}_3 + \vec{v}\cdot\nabla\vec{v} - (Re)^{-1}\Delta\vec{v} = 0$$

$$\frac{\partial\rho}{\partial t} - (\Gamma)^{-1}\ (Fr)^{-2}\ w + \vec{v}\cdot\nabla\rho - (Re)^{-1}(Pr)^{-1}\Delta\rho = 0 \tag{2.3}$$

$$div\ \vec{v} = 0$$

where the non-dimensional numbers are

$$
\begin{aligned}
&Ro = \frac{U}{Lf} \quad \text{Rossby number}, && Fr = \frac{U}{LN} \quad \text{Froude number}, \\
&\overline{P} = \frac{p}{\rho_b U^2} \quad \text{Euler number}, && Re = \frac{\rho_b U L}{\mu} \quad \text{Reynolds number}, \\
&Pr = \frac{\mu}{\rho_b D} \quad \text{Prandtl number}, && \Gamma = \frac{BgL}{U^2} \ .
\end{aligned}
\tag{2.4}
$$

It is well known that for incompressible flows the pressure ϕ is not an independent physical variable. In fact, the pressure plays role of a Lagrange multiplier to enforce the incompressibility constraint. The pressure ϕ can be eliminated from the Boussinesq equations in standard fashion, by computing the divergence of the momentum equation in (2.3) together with the incompressibility constraint, and then solving the resulting elliptic equation for the pressure ϕ to get

$$
\overline{P}\,\nabla\phi = \nabla\Delta^{-1}\left((Ro)^{-1}\vec{\eta}\cdot\vec{\omega} - \Gamma\frac{\partial\rho}{\partial x_3} - \ div\ (\vec{v}\cdot\nabla\vec{v})\right) \ . \tag{2.5}
$$

Introducing Eq. (2.5) for the pressure back into the Boussinesq equations in Eq. (2.3) yields

Nonlocal Rotating Boussinesq Equations with Slanted Rotation

$$
\begin{aligned}
\frac{\partial\vec{v}}{\partial t} + (Ro)^{-1}\vec{\eta}\times\vec{v} + \Gamma\rho\vec{e}_3 + \nabla\Delta^{-1}\left((Ro)^{-1}\vec{\eta}\cdot\vec{\omega} - \Gamma\frac{\partial\rho}{\partial x_3}\right) + \\
\vec{v}\cdot\nabla\vec{v} - \nabla\Delta^{-1}\left(\ div\ (\vec{v}\cdot\nabla\vec{v})\right) - (Re)^{-1}\Delta\vec{v} = 0
\end{aligned}
\tag{2.6}
$$

$$
\frac{\partial\rho}{\partial t} - (\Gamma)^{-1}\,(Fr)^{-2}\,w + \vec{v}\cdot\nabla\rho - (Re)^{-1}(Pr)^{-1}\Delta\rho = 0 \ .
$$

This reduced form of the Boussinesq equations is needed for the application of the theory developed below. The equations in (2.6) can be written in abstract form as

$$
\frac{\partial\vec{u}}{\partial t} + \mathcal{L}(\vec{u}) + \mathcal{B}(\vec{u},\vec{u}) - \mathcal{D}(\vec{u}) = 0 \tag{2.7}
$$

where is $\vec{u}$ given by

$$
\vec{u} = \begin{pmatrix} \vec{v} \\ \rho \end{pmatrix},
$$

the linear operator $\mathcal{L}(\vec{u})$ is given by

$$\mathcal{L}(\vec{u}) = \begin{pmatrix} (Ro)^{-1}\vec{\eta} \times \vec{v} + \Gamma\rho\vec{e}_3 + \nabla\Delta^{-1}\left((Ro)^{-1}\vec{\eta}\cdot\vec{\omega} - \Gamma\frac{\partial\rho}{\partial x_3}\right) \\ -(\Gamma)^{-1}\,(Fr)^{-2}\,w \end{pmatrix}, \tag{2.8}$$

the quadratic operator $\mathcal{B}(\vec{u}, \vec{u})$ is given by

$$\mathcal{B}(\vec{u}, \vec{u}) = \begin{pmatrix} \vec{v}\cdot\nabla\vec{v} - \nabla\Delta^{-1}\,(\,div\,(\vec{v}\cdot\nabla\vec{v})) \\ \vec{v}\cdot\nabla\rho \end{pmatrix}, \tag{2.9}$$

and the diffusion operator $\mathcal{D}(\vec{u})$ is given by

$$\mathcal{D}(\vec{u}) = \begin{pmatrix} (Re)^{-1}\Delta\vec{v} \\ (Re)^{-1}(Pr)^{-1}\Delta\rho \end{pmatrix}. \tag{2.10}$$

We remark that the systems given in Eqs. (2.3) and (2.6) are equivalent under the natural assumption of incompressibility for the initial velocity field data $\vec{v}$. In fact, it is straightforward to check that $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{D}$ are divergence free. Therefore taking the divergence of Eq. (2.7) gives

$$\frac{\partial}{\partial t}\,div\,\vec{v} = 0$$

which shows that the velocity $\vec{v}$ is divergence free for all times provided it is divergence free initially.

2.2 The Potential Vorticity Equation

A fundamental quantity in geophysical fluid mechanics is the potential vorticity q. Here it is defined as the product of the absolute vorticity $\vec{\omega}_a$ and the gradient of the density $\tilde{\rho}$, $q = \vec{\omega}_a \cdot \nabla\tilde{\rho}$, where the density $\tilde{\rho}$ is given by Eq. (2.2) and the absolute vorticity $\vec{\omega}_a = \vec{\omega} + \vec{\Omega}$, with $\vec{\omega} =$ *curl* $\vec{v}$, and $\vec{\Omega} = f\vec{\eta}$. If there are no viscous or mass diffusion effects in the rotating Boussinesq equations in (2.1), then Ertel's theorem (Pedlosky, 1987) shows the fundamental fact that the potential vorticity q is conserved along particle trajectories of the flow. On the other hand, if the diffusive effects are kept in Eq. (2.1), then the same vector calculus manipulations utilized for the proof of Ertel's theorem yield the following equation for the potential vorticity q

$$\frac{Dq}{Dt} = \frac{D}{Dt}(\vec{\omega}_a \cdot \nabla\tilde{\rho}) = \mu\rho_b^{-1}\Delta\vec{\omega}\cdot\nabla\tilde{\rho} + D\Delta\nabla\rho\cdot\vec{\omega}_a\,, \tag{2.11}$$

and in non-dimensional form the above equation reduces to the

Potential vorticity equation

$$\frac{D}{Dt}\left[\omega_3 - (Ro)^{-1}\Gamma(Fr)^2(\nabla\rho\cdot\vec{\eta}) - \Gamma\,(Fr)^2(\vec{\omega}\cdot\nabla\rho)\right] =$$

$$(Re)^{-1}\Delta\left[\omega_3 \; - (Pr)^{-1}(Ro)^{-1}\Gamma(Fr)^2(\nabla\rho\cdot\vec{\eta})\right] \tag{2.12}$$

$$- (Re)^{-1}\Gamma(Fr)^2\left[\Delta\vec{\omega}\cdot\nabla\rho + (Pr)^{-1}\nabla\Delta\rho\cdot\vec{\omega}\right],$$

where $\vec{\omega} = \; curl\,\vec{v} = (\omega_1, \omega_2, \omega_3)$ is the vorticity.

2.3 Scalings for the Rotating Boussinesq Equation

Simplifications in the geophysical fluid equations can be achieved because of the disparity in the space-time scales associated with the various physical processes taking place in the fluid. For example, in many mesoscale or large scale flows in the atmosphere and the ocean the characteristic frequency U/L associated with the convective motion is substantially smaller than the buoyancy frequency N of the fluid; this implies that the Froude number $Fr = U/LN$ is small. In addition, for large scale flows the characteristic frequency U/L is often smaller than the rotation frequency f; in this case the Rossby number $Ro = U/Lf$ is also small. On the other hand, for many flows on mesoscales, the Rossby number is not necessarily small. Therefore it is natural to consider both the asymptotic regime of small Fr but finite Ro numbers, as well as the quasigeostrophic regime with small Fr and Ro numbers. Related scalings have been utilized previously (Lilly, 1983).

First we consider the asymptotic limit of small Froude and finite Rossby number. In this case the non-dimensional numbers in Eq. (2.4) are related as follows

Low Froude, finite Rossby scaling

$$\begin{aligned} &Fr = \epsilon, \text{ with } \quad \epsilon \ll 1 \\ &Ro = O(1), \text{ as } \quad \epsilon \to 0 \\ &\overline{P} = \overline{P}_0\epsilon^{-1} \\ &\Gamma = \epsilon^{-1} \\ &Re = O(1), \text{ as } \quad \epsilon \to 0 \\ &Pr = O(1), \text{ as } \quad \epsilon \to 0\,. \end{aligned} \tag{2.13}$$

With this scaling the rotating Boussinesq equations in (2.7) take the form

$$\begin{aligned} &\frac{\partial\vec{u}}{\partial t} + \epsilon^{-1}\mathcal{L}_F(\vec{u}) + \mathcal{L}_S(\vec{u}) + \mathcal{B}(\vec{u},\vec{u}) - \mathcal{D}(\vec{u}) = 0 \\ &\vec{u}|_{t=0} = \vec{u}_0(x)\,. \end{aligned} \tag{2.14}$$

Here the linear operator $\mathcal{L} = \epsilon^{-1}\mathcal{L}_F + \mathcal{L}_S$ in Eq. (2.7) decomposes into a fast component $\mathcal{L}_F$ and a slow component $\mathcal{L}_S$, with the fast component $\mathcal{L}_F$ given by

$$\mathcal{L}_F(\vec{u}) = \begin{pmatrix} \rho\vec{e}_3 - \nabla\Delta^{-1}\dfrac{\partial\rho}{\partial x_3} \\ \\ -w \end{pmatrix} \tag{2.15}$$

and the slow component $\mathcal{L}_S$ given by

$$\mathcal{L}_S(\vec{u}) = \begin{pmatrix} (Ro)^{-1}\left\{\vec{\eta}\times\vec{v} + \nabla\Delta^{-1}(\vec{\eta}\cdot\vec{\omega})\right\} \\ \\ 0 \end{pmatrix} . \tag{2.16}$$

The formulas for the bilinear operator $\mathcal{B}(\vec{u},\vec{u})$ and the diffusion operator $\mathcal{D}(\vec{u})$ are still given by Eqs. (2.9) and (2.10) respectively. Finally, with the small Fr, finite Ro scaling, the potential vorticity equation in (2.12) becomes

$$\begin{aligned} \frac{D}{Dt}\left[\omega_3 - \epsilon\left((Ro)^{-1}(\nabla\rho\cdot\vec{\eta}) + \vec{\omega}\cdot\nabla\rho\right)\right] &= (Re)^{-1}\Delta\omega_3 \\ &- \epsilon(Re)^{-1}\left[\Delta\vec{\omega}\cdot\nabla\rho + (Pr)^{-1}\nabla\Delta\rho\cdot\vec{\omega} + (Pr)^{-1}(Ro)^{-1}\nabla\rho\cdot\vec{\eta}\right] . \end{aligned} \tag{2.17}$$

Next we consider the quasigeostrophic limit of small Froude and small Rossby numbers. In this case the scaling for the nondimensional numbers in Eq. (2.4) is given by

Low Froude, low Rossby scaling

$$\begin{aligned} &Ro = \epsilon F^{-1}, \text{ with } \quad \epsilon \ll 1 \\ &Fr = \epsilon \\ &\overline{P} = \overline{P}_0\epsilon^{-1} \\ &\Gamma = (Fr)^{-1} = \epsilon^{-1} \\ &Re \geq O(1), \text{ as } \quad \epsilon \to 0 \\ &Pr, \text{ fixed as } \quad \epsilon \to 0 . \end{aligned} \tag{2.18}$$

With this scaling the rotating Boussinesq equations in (2.7) have the same form as Eq. (2.15)

$$\begin{aligned} &\frac{\partial\vec{u}}{\partial t} + \epsilon^{-1}\mathcal{L}_F(\vec{u}) + \mathcal{B}(\vec{u},\vec{u}) - \mathcal{D}(\vec{u}) = 0 \\ &\vec{u}|_{t=0} = \vec{u}_0(x) , \end{aligned} \tag{2.19}$$

except that now the linear operator $\mathcal{L} = \epsilon^{-1}\mathcal{L}_F$ has only a fast component $\mathcal{L}_F$ ($\mathcal{L}_S = 0$)

$$\mathcal{L}_F(\vec{u}) = \begin{pmatrix} F\vec{\eta} \times \vec{v} + \rho\vec{e}_3 + \nabla\Delta^{-1}\left(F\vec{\eta}\cdot\vec{\omega} - \dfrac{\partial\rho}{\partial x_3}\right) \\ -w \end{pmatrix} \tag{2.20}$$

where the bilinear operator $\mathcal{B}(\vec{u},\vec{u})$ and the diffusion operator $\mathcal{D}(\vec{u})$ are again given by Eqs. (2.9) and (2.10). On the other hand, in the low Fr and low Ro regime, the potential vorticity equation in (2.12) becomes

$$\begin{aligned} \frac{D}{Dt}\left[\omega_3 - F(\nabla\rho\cdot\vec{\eta}) - \epsilon\,\vec{\omega}\cdot\nabla\rho\right] &= (Re)^{-1}\Delta\left[\omega_3 \; - (Pr)^{-1}F(\nabla\rho\cdot\vec{\eta})\right] \\ &\quad - \epsilon(Re)^{-1}\left[\Delta\vec{\omega}\cdot\nabla\rho + (Pr)^{-1}\nabla\Delta\rho\cdot\vec{\omega}\right] . \end{aligned} \tag{2.21}$$

In both the asymptotic limit of small Fr and finite Ro numbers, as well as for small Fr and small Ro numbers, the scaling results in the introduction of the singular term $\epsilon^{-1}\mathcal{L}_F$, and in the appearance of separated fast and slow scales of motion. This can be shown clearly in the linear analysis of the Bousinessq equations without diffusion. Specifically, consider the linear equation associated with either (2.14) or (2.19)

$$\frac{\partial\vec{u}}{\partial t} + \epsilon^{-1}\mathcal{L}_F(\vec{u}) = 0 \tag{2.22}$$

with periodic boundary conditions. The periodic eigenfunctions

$$\vec{u} = \exp(i\vec{k}\cdot\vec{x} - i\omega(\vec{k})t)\vec{r} \tag{2.23}$$

of the above equation that satisfy the incompressibility condition are either slow (vortical) modes with frequency $\omega(\vec{k}) = 0$, or fast (gravity) waves with $\omega(\vec{k}) = \epsilon^{-1}\Omega(\vec{k}) \neq 0$. For example, the dispersion relation $\Omega = \Omega(\vec{k})$ in the small Fr and finite Ro regime is given by

$$\Omega = \pm|\vec{k}_H|/|\vec{k}| \tag{2.24}$$

and for the small Fr and small Ro number it is given by

$$\Omega = \pm\left(F^2(\vec{\eta}\cdot\vec{k})^2 + |\vec{k}_H|^2\right)^{1/2}/|\vec{k}| \, . \tag{2.25}$$

The concrete formulas for the corresponding periodic eigenfunctions will be given in detail in Secs. 3 and 4. Therefore the equations admit slow modes moving on time scales of order $O(1)$ and fast waves moving on time scales of order $O(\epsilon^{-1})$. This separation of scales will be exploited below to derive averaged equations valid on order one advective time scales for the solutions of the Boussinesq equations in (2.14) and (2.19) in the singular asymptotic limit where the Froude number $Fr = \epsilon$ goes to zero.

2.4 Energy Conservation Principle

Next we consider the principle of conservation of energy for the solutions of the Boussinesq equations for the asymptotic regime of small Fr and finite Ro numbers, and the quasigeostrophic regime with small Fr and Ro numbers. In both cases the Boussinesq equations have the structure given by

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + \epsilon^{-1}\mathcal{L}_F(\vec{u}) + \mathcal{L}_S(\vec{u}) + \mathcal{B}(\vec{u}.\vec{u}) - \mathcal{D}(\vec{u}) = 0 \\ &\vec{u}|_{t=0} = \vec{u}_0(x) \end{aligned} \tag{2.26}$$

with $div\ \vec{v}_0 = 0$ initially. The linear operator $\mathcal{L} = \epsilon^{-1}\mathcal{L}_F + \mathcal{L}_S$ is given by Eqs. (2.15) and (2.16) for the small Fr, finite Ro limit, and by Eq. (2.20) in the small Fr, small Ro limit. A straightforward calculation shows that the operators $\mathcal{L}_F$ and $\mathcal{L}_S$ are skew-hermitian on the space of spatially periodic functions $\vec{u}(\vec{x}) =^t (\vec{v}(\vec{x})\ ,\ \rho(\vec{x}))$ with divergence zero velocity field $\vec{v}$. This immediately implies that the linear equation (2.22) conserves energy. Moreover, a direct calculation with the nonlinear equation in (2.26), with the bilinear operator $\mathcal{B}(\vec{u},\vec{u})$ and the diffusion operator $\mathcal{D}(\vec{u})$ given in Eqs. (2.9) and (2.10), shows that the sum of the kinetic and potential energies satisfies the identity

$$\frac{1}{2}\frac{d}{dt}\int |\vec{v}|^2 + |\rho|^2 dx = -\int (Re)^{-1}|\nabla\vec{v}|^2 + (Re)^{-1}(Pr)^{-1}|\nabla\rho|^2\, dx\ . \tag{2.27}$$

Therefore the energy is non-increasing in the presence of diffusion, and conserved otherwise. More importantly, the identity in Eq. (2.27) is free of the singular parameter ϵ in Eq. (2.26), and therefore it is valid uniformly in ϵ. Energy estimates similar to Eq. (2.27) can also be derived for higher order spatial derivatives of the solution $\vec{u}$, which are also independent of the small parameter ϵ. Once we have energy bounds on the function $\vec{u}$ and its space derivatives up to order s (in practice the minimum order s needed in 3-D is $s = 3$), then standard arguments (Majda, 1984) shows that the solutions $\vec{u}$ of Eq. (2.26) exist in a common time interval $[0, T]$ *uniformly* in $0 < \epsilon \leq 1$, and the energy norm of $\vec{u}$ and its derivatives up to order s are uniformly bounded in ϵ. The question now is to deduce what is the limiting form of the solutions in the asymptotic limit when $\epsilon \to 0$.

Next we derive the averaged equations for the singular limit of the Boussinesq equations in (2.26) in the asymptotic limit of $\epsilon \to 0$. The derivation is based on the method of multiple scales, where the solution $\vec{u}^{\epsilon}$ of Eq. (2.26) depends on the fast time scale $\tau = t/\epsilon$, and on the slow time scale t, and that for $\epsilon \ll 1$ the solution $\vec{u}^{\epsilon}$ has the expansion

$$\vec{u}^{\epsilon}(\vec{x},t,\tau) = \vec{u}^0(\vec{x},t,\tau)\Big|_{\tau=t/\epsilon} + \epsilon\vec{u}^1(\vec{x},t,\tau)\Big|_{\tau=t/\epsilon}. \tag{2.28}$$

In order to avoid secular growth of the first order term $\vec{u}^1$, we require that $\vec{u}^1$ satisfies the sublinear growth condition $|\vec{u}^1(\vec{x},t,\tau)| = o(\tau)$ uniformly on $0 \le \tau \le T/\epsilon$. We substitute the expansion in (2.28) into Eq. (2.26) and collect powers of ϵ. To order $O(\epsilon^{-1})$ we obtain the equation

$$\frac{\partial \vec{u}^0}{\partial \tau} + \mathcal{L}_F(\vec{u}^0) = 0 \tag{2.29}$$

with the solution

$$\vec{u}^0(\vec{x},t,\tau) = e^{-\tau\mathcal{L}_F}\bar{u}(\vec{x},t). \tag{2.30}$$

The next order $O(\epsilon^0)$ yields

$$\frac{\partial \vec{u}^1}{\partial \tau} + \mathcal{L}_F(\vec{u}^1) = -\left(\frac{\partial \vec{u}^0}{\partial t} + \mathcal{L}_S(\vec{u}^0) + \mathcal{B}(\vec{u}^0,\vec{u}^0) - \mathcal{D}(\vec{u}^0)\right) \tag{2.31}$$

with $\vec{u}^0$ given by Eq. (2.30). The solution of Eq. (2.31) is given by Duhamel's formula

$$\begin{aligned} e^{\tau\mathcal{L}_F}\vec{u}^1 &= \vec{u}^1(\vec{x},t,\tau)\Big|_{\tau=0} - \tau\frac{\partial \bar{u}}{\partial t}(\vec{x},t) - \\ &\int_0^{\tau} e^{s\mathcal{L}_F}\left(\mathcal{L}_S(e^{-s\mathcal{L}_F}\bar{u}) + \mathcal{B}(e^{-s\mathcal{L}_F}\bar{u}, e^{-s\mathcal{L}_F}\bar{u}) - \mathcal{D}(e^{-s\mathcal{L}_F}\bar{u})\right)\, ds. \end{aligned} \tag{2.32}$$

Thus, in order to satisfy the sublinear growth condition below Eq. (2.28) and avoid leading order secularity, the state $\bar{u}(\vec{x},t)$ has to solve the

Fast wave averaging equation

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t}(\vec{x},t) = -\lim_{T\to\infty}\frac{1}{T}\int_0^{T} e^{s\mathcal{L}_F}\Big(&\mathcal{L}_S(e^{-s\mathcal{L}_F}\bar{u}) + \mathcal{B}(e^{-s\mathcal{L}_F}\bar{u}, e^{-s\mathcal{L}_F}\bar{u}) \\ &-\mathcal{D}(e^{-s\mathcal{L}_F}\bar{u})\Big)\, ds \end{aligned} \tag{2.33}$$

$$\bar{u}(\vec{x},t)\big|_{t=0} = \vec{u}_0(\vec{x}) \; .$$

We conclude that to leading order in ϵ the solution $\vec{u}^{\epsilon}(\vec{x},t)$ is given by

$$\vec{u}^{\epsilon}(\vec{x},t) = e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}(\vec{x},t) + o(1) \tag{2.34}$$

where $\bar{u}(\vec{x},t)$ satisfies the averaged equation (2.33). This completes the formal derivation of the averaged equations for the reduced dynamics.

Although the asymptotic derivation presented here is formal, the mathematical validity of the approximation in Eqs. (2.33) and (2.34) has been established with complete rigor by Embid and Majda (1996); in essence the proof combines a "cancelation of oscillations" argument introduced earlier by Schochet (1994) and motivated by the above perturbation argument with standard techniques developed in the study of singular limits (Klainerman and Majda, 1981; Majda, 1984) together with the energy estimates available for the Boussinesq equations. The explicit form of the reduced averaged equations in (2.33) will be given in terms of expansions in the eigenfunctions of the operator $\mathcal{L}_F$, and ultimately it will involve three wave resonant contributions from slow and/or fast waves. This will be carried out in the low Fr, finite Ro limit in Sec. 3, and for the low Fr, low Ro limit in Sec. 4.

The leading term of the solution $\vec{u}^{\epsilon}(\vec{x},t)$ in Eq. (2.34) involves both fast and slow modes. In fact, since $\mathcal{L}_F$ is skew-hermitian, then $\bar{u}(\vec{x},t)$ has an orthogonal decomposition in terms of a slow component $\bar{u}^S$ and a fast component $\bar{u}^F$

$$\bar{u}(\vec{x},t) = \bar{u}^S(\vec{x},t) + \bar{u}^F(\vec{x},t) \tag{2.35}$$

with $\mathcal{L}_F(\bar{u}^S) = 0$. With this decomposition of $\bar{u}$, the solution $\vec{u}^{\epsilon}$ in Eq. (2.34) takes the form

$$\vec{u}^{\epsilon}(\vec{x},t) = e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}(\vec{x},t) = \bar{u}^S(\vec{x},t) + e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}^F(\vec{x},t) + o(1) \ , \tag{2.36}$$

which shows that $\bar{u}^S$ evolves on the slow scale alone whereas $\bar{u}^F$ has all the fast oscillations. A special case of the averaged equations arises if we demand that the solution evolves only on the slow scale. In this case we must impose the constraint $\bar{u} = \bar{u}^S$, so that the leading term in eq. (2.34) has no fast oscillations. If we let P be the orthogonal projection onto the nullspace $N(\mathcal{L}_F)$, then the above constraint translates as $\mathcal{L}_F(\bar{u}) = 0$, and the averaged equation in (2.33) reduces to the

Slow dynamics equations

$$\begin{aligned} &\frac{\partial \bar{u}}{\partial t} + P(\mathcal{L}_S(\bar{u}) + \mathcal{B}(\bar{u},\bar{u}) - \mathcal{D}(\bar{u})) = 0 \\ &\bar{u}(\vec{x},0) = \vec{u}_0(\vec{x}) \in N(\mathcal{L}_F) \ . \end{aligned} \tag{2.37}$$

The restriction $\mathcal{L}_F(\vec{u}_0) = 0$ requires that there are no fast waves present initially and it is sufficient to guarantee that the solution $\bar{u}$ will satisfy $\mathcal{L}_F(\bar{u}) = 0$ for later times.

From Eq. (2.15) follows that in the low Fr, finite Ro regime, the restriction $\mathcal{L}_F(\vec{u}) = 0$ is equivalent to *hydrostatic balance and strong stratification*

$$\begin{aligned} \rho\vec{e}_3 - \nabla\Delta^{-1}\frac{\partial\rho}{\partial x_3} = 0 \\ w = 0 \end{aligned} \tag{2.38}$$

and Eq. (2.20) shows that in the low Fr, low Ro regime, the restriction corresponds to *geostrophic-hydrostatic balance and strong stratification*

$$\begin{aligned} F\vec{\eta}\times\vec{v} + \rho\vec{e}_3 + \nabla\Delta^{-1}\left(F\vec{\eta}\cdot\vec{\omega} - \frac{\partial\rho}{\partial x_3}\right) = 0 \\ w = 0 \end{aligned} \tag{2.39}$$

Notice in particular that when the ratio $F = Fr/Ro$ is zero, then Eq. (2.39) reduces to Eq. (2.38). The concrete formulas for the slow dynamics equations for both asymptotic limits will be given in Secs. 3 and 4.

In general, if the initial data has gravity waves, then the solution $\vec{u}^{\epsilon}$ in Eq. (2.34) will have fast and slow modes. Because of the fast oscillations the solution $\vec{u}^{\epsilon}$ *cannot* converge in the strong sense. However, it can be shown (Embid and Majda, 1996) that the solution $\vec{u}^{\epsilon}$ in Eq. (2.36) converges weakly to $\bar{u}^S$, and that $\bar{u}^S$ satisfies the potential vorticity equation. This implies that the slow vortical modes evolve through their own dynamics, but can interact as a catalyzer to promote energy exchanges among the fast gravity waves. This point of view has also been emphasized recently by Bartello (1995) in an interesting study.

3. LOW FROUDE NUMBER LIMITING DYNAMICS AT FINITE ROSSBY NUMBERS

Here we consider the equations for the limiting dynamics of the rotating Boussinesq equations at low Froude number and finite Rossby number. First we carry out the spectral analysis of the linear operator $\mathcal{L}_F$ and utilize the resulting basis of Fourier eigenfunctions to derive concrete formulas for the fast averaged equations in (2.33) in terms of the Fourier amplitudes. Afterwards we will analyze the resulting limiting dynamics for the slow and the fast waves.

3.1 Fourier Analysis of the Fast Operator $\mathcal{L}_F$

Here we study the concrete form of the averaged equations in (2.33) for the dynamics of the rotating Boussinesq equations in the limit of small Froude number and finite Rossby number. For simplicity we assume periodic boundary conditions in the spatial domain, and utilize the Fourier eigenfunctions of the operator $\mathcal{L}_F$ given by Eq. (2.15). The periodic eigenfunctions are of the form

$$\vec{u} = \exp(i\vec{k}\cdot\vec{x} - i\omega(\vec{k})t)\vec{r} \tag{3.1}$$

where $\vec{u}(\vec{x},t)$ must satisfy the incompressibility condition. Straightforward algebra shows that the eigenfrequencies $\omega(\vec{k})$ associated to the wave number $\vec{k}$ are

$$\omega_{(\vec{k})}^{(-1)} = -\frac{|\vec{k}_H|}{|\vec{k}|}, \qquad \omega_{(\vec{k})}^{(0)} = 0, \text{ (double)}, \qquad \omega_{(\vec{k})}^{(1)} = \frac{|\vec{k}_H|}{|\vec{k}|}. \tag{3.2}$$

when $\vec{k} \neq 0$, and for $\vec{k} = 0$ they are

$$\omega_{(0)}^{(-1)} = -1, \qquad \omega_{(0)}^{(0)} = 0, \text{ (double)}, \qquad \omega_{(0)}^{(1)} = 1. \tag{3.3}$$

The associated right eigenvectors $\vec{r}$ in Eq. (3.1) are given as follows. In the first case when $\vec{k}_H \neq 0$ then the eigenvectors are

$$\begin{matrix} \vec{r}_{(\vec{k})}^{(-1)} & \vec{r}_{(\vec{k})}^{(1)} & \vec{r}_{(\vec{k})}^{(0)} \\ \begin{pmatrix} -\frac{i}{\sqrt{2}}\frac{k_1k_3}{|\vec{k}_H||\vec{k}|} \\ -\frac{i}{\sqrt{2}}\frac{k_2k_3}{|\vec{k}_H||\vec{k}|} \\ \frac{i}{\sqrt{2}}\frac{|\vec{k}_H|}{|\vec{k}|} \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} \frac{i}{\sqrt{2}}\frac{k_1k_3}{|\vec{k}_H||\vec{k}|} \\ \frac{i}{\sqrt{2}}\frac{k_2k_3}{|\vec{k}_H||\vec{k}|} \\ -\frac{i}{\sqrt{2}}\frac{|\vec{k}_H|}{|\vec{k}|} \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} -i\frac{k_2}{|\vec{k}_H|} \\ i\frac{k_1}{|\vec{k}_H|} \\ 0 \\ 0 \end{pmatrix} \end{matrix} \tag{3.4}$$

where the fourth eigenvector has been discarded because the corresponding eigensolution does not yield an incompressible velocity field. The eigenfunctions associated with $\vec{r}_{(\vec{k})}^{(\pm 1)}$ represent fast gravity waves, and the one associated with $\vec{r}_{(\vec{k})}^{(0)}$ represents a slow vortical mode. In the second case when $\vec{k}_H = 0$ then all the eigenfrequencies $\omega(\vec{k})$ are zero. In this case the three eigenvectors are associated with slow waves and are given by

$$\begin{array}{ccc} \vec{r}^{\,(-1)}_{\,(\vec{k})} & \vec{r}^{\,(0)}_{\,(\vec{k})} & \vec{r}^{\,(1)}_{\,(\vec{k})} \\ \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \end{array}. \tag{3.5}$$

Finally, the case of $\vec{k} = 0$ corresponds to mean flows. In this case the incompressibility condition is trivially satisfied and the four eigenvectors provide valid eigenfunction solutions

$$\begin{array}{cccc} \vec{r}^{\,(-1)}_{\,(0)} & \vec{r}^{\,(1)}_{\,(0)} & \vec{r}^{\,(0)}_{\,(0)} & \tilde{\vec{r}}^{\,(0)}_{\,(0)} \\ \begin{pmatrix} 0 \\ 0 \\ \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{array}. \tag{3.6}$$

The first two eigenvectors $\vec{r}^{\,(\pm1)}_{\,(0)}$ correspond to fast waves moving with the buoyancy frequency. The other two eigenvectors $\vec{r}^{\,(0)}_{\,(0)}$ and $\tilde{\vec{r}}^{\,(0)}_{\,(0)}$ are associated with slow modes. This concludes the spectral analysis of the operator $\mathcal{L}_F$. We also remark that the eigenvectors in Eq. (3.4), Eq. (3.5) and Eq. (3.6) satisfy the symmetry relations

$$\overline{\vec{r}^{\,(\alpha)}_{\,(\vec{k})}} = \vec{r}^{\,(-\alpha)}_{\,(-\vec{k})}, \quad \text{for} \quad \alpha = -1, 0, 1 \tag{3.7}$$

3.2 Concrete Form of the Fast Averaged Equations

To compute the concrete form of the averaged equations in (2.33) we expand the solution $\bar{u}(\vec{x}, t)$ in terms of the periodic eigenfunctions of $\mathcal{L}_F$ discussed earlier

$$\bar{u}(\vec{x}, t) = \sum_{\vec{k}\in Z^3} \sum_{\alpha=-1}^{1} e^{i\vec{k}\cdot\vec{x}} \sigma^{\,(\alpha)}_{\,(\vec{k})}(t) \vec{r}^{\,(\alpha)}_{\,(\vec{k})}, \tag{3.8}$$

where in order to insure that the $\bar{u}(\vec{x}, t)$ is real valued the amplitudes $\sigma^{(\alpha)}_{(\vec{k})}(t)$ must satisfy the symmetry relations

$$\overline{\sigma^{(\alpha)}_{(\vec{k})}(t)} = \sigma^{(-\alpha)}_{(-\vec{k})}(t) \ . \tag{3.9}$$

In terms of this representation the exponential $e^{-s\mathcal{L}_F}\bar{u}(\vec{x}, t)$ is given by

$$e^{-s\mathcal{L}_F}\bar{u}(\vec{x}, t) = \sum_{\vec{k}\in Z^3} \sum_{\alpha=-1}^{1} e^{i\left(\vec{k}\cdot\vec{x} - s\omega^{(\alpha)}_{(\vec{k})}\right)} \sigma^{(\alpha)}_{(\vec{k})} \vec{r}^{(\alpha)}_{(\vec{k})}. \tag{3.10}$$

Next we have to introduce this representation of the exponential into the bilinear form $\mathcal{B}$, the linear term $\mathcal{L}_S$ and the diffusive term $\mathcal{D}$ in the right hand side of the averaged equation (2.33). Introducing Eq. (3.10) into the bilinear form $\mathcal{B}$ in Eq. (2.33) and expanding the result yields

$$\begin{aligned} \mathcal{B}(e^{-s\mathcal{L}_F}\bar{u}, e^{-s\mathcal{L}_F}\bar{u}) = \sum_{\vec{k}\in Z^3} \sum_{\alpha=-1}^{1} \Bigg\{ \sum_{\vec{k}'+\vec{k}''=\vec{k}} \sum_{\alpha',\alpha''=-1}^{1} e^{i\left(\vec{k}\cdot\vec{x} - s\left(\omega^{(\alpha')}_{(\vec{k}')} + \omega^{(\alpha'')}_{(\vec{k}'')}\right)\right)} \times \\ B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})} \sigma^{(\alpha')}_{(\vec{k}')}(t) \sigma^{(\alpha'')}_{(\vec{k}'')}(t) \Bigg\} \vec{r}^{(\alpha)}_{(\vec{k})}, \end{aligned} \tag{3.11}$$

where the interaction coefficient $B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})}$ is given explicitly by

$$\begin{aligned} B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})} &= \frac{i}{2} \left[(\vec{v}^{(\alpha')}_{(\vec{k}')} \cdot \vec{k}'') \langle \vec{r}^{(\alpha'')}_{(\vec{k}'')}, \vec{r}^{(\alpha)}_{(\vec{k})} \rangle + (\vec{v}^{(\alpha'')}_{(\vec{k}'')} \cdot \vec{k}') \langle \vec{r}^{(\alpha')}_{(\vec{k}')}, \vec{r}^{(\alpha)}_{(\vec{k})} \rangle \right] \quad \text{if } \vec{k} \neq 0 \\ B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})} &= \frac{i}{2} \left[(\vec{v}^{(\alpha')}_{(\vec{k}')} \cdot \vec{k}'') \rho^{(\alpha'')}_{(\vec{k}'')} + (\vec{v}^{(\alpha'')}_{(\vec{k}'')} \cdot \vec{k}') \rho^{(\alpha')}_{(\vec{k}')} \right] \rho^{(\alpha)}_{(\vec{k})} \quad \text{if } \vec{k} = 0 \ . \end{aligned} \tag{3.12}$$

Therefore the quadratic contributions due to $\mathcal{B}$ in the averaged equations are given by

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \int_0^T & e^{s\mathcal{L}} \mathcal{B}(\vec{x}, t, e^{-s\mathcal{L}}\vec{u}(\vec{x}, t), e^{-s\mathcal{L}}\vec{u}(\vec{x}, t))\, ds = \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{\vec{k}\in Z^3} \sum_{\alpha=-1}^{1} \Bigg\{ \sum_{\vec{k}'+\vec{k}''=\vec{k}} \sum_{\alpha',\alpha''=-1}^{1} e^{i\left(\vec{k}\cdot\vec{x} - \left(\omega^{(\alpha')}_{(\vec{k}')} + \omega^{(\alpha'')}_{(\vec{k}'')} - \omega^{(\alpha)}_{(\vec{k})}\right)s\right)} \times \\ & \times B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})} \sigma^{(\alpha')}_{(\vec{k}')}(t) \sigma^{(\alpha'')}_{(\vec{k}'')}(t) \Bigg\} \vec{r}^{(\alpha)}_{(\vec{k})}\, ds \ . \end{aligned} \tag{3.13}$$

Clearly the only non-zero contributions left after averaging are those that come from *three wave resonant interactions*

$$\vec{k}' + \vec{k}'' = \vec{k}$$

$$\omega^{(\alpha')}_{(\vec{k}')} + \omega^{(\alpha'')}_{(\vec{k}'')} = \omega^{(\alpha)}_{(\vec{k})} \,. \tag{3.14}$$

Therefore the quadratic contributions in Eq. (3.13) reduce to

$$\sum_{\vec{k}\in Z^3} \sum_{\alpha=-1}^{1} \left\{ \sum_{(\vec{k}',\vec{k}'',\alpha',\alpha'')\in S_{\alpha,\vec{k}}} B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})} \sigma^{(\alpha')}_{(\vec{k}')}(t) \sigma^{(\alpha'')}_{(\vec{k}'')}(t) \right\} e^{i\vec{k}\cdot\vec{x}} \vec{r}^{\,(\alpha)}_{(\vec{k})} \tag{3.15}$$

where $S_{\alpha,\vec{k}}$ is the set associated with all three wave resonances

$$S_{\alpha,\vec{k}} = \left\{ (\vec{k}', \vec{k}'', \alpha', \alpha'') |\ \vec{k}' + \vec{k}'' = \vec{k},\ \omega^{(\alpha')}_{(\vec{k}')} + \omega^{(\alpha'')}_{(\vec{k}'')} = \omega^{(\alpha)}_{(\vec{k})} \right\}. \tag{3.16}$$

The contributions from the linear operator $\mathcal{L}_S$ and the diffusion operator $\mathcal{D}$ are simpler. Since these operators are linear, the only contributions from averaging in time are the result of modes with the same frequency and wave number, $\omega^{(\alpha)}_{(\vec{k})} = \omega^{(\alpha')}_{(\vec{k})}$. Therefore the contribution from $\mathcal{L}_S$ to the averaged equations is

$$\begin{aligned} \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} & e^{s\mathcal{L}_F} \mathcal{L}_S \left(e^{-s\mathcal{L}_F} \vec{u}(\vec{x}, t) \right) ds = \\ & = \sum_{\vec{k}\in Z^3} \sum_{\omega^{(\alpha')}_{(\vec{k})} = \omega^{(\alpha)}_{(\vec{k})}} L^{(\alpha',\alpha)}_{(\vec{k})} \sigma^{(\alpha')}_{(\vec{k})}(t) e^{i\vec{k}\cdot\vec{x}} \vec{r}^{\,(\alpha)}_{(\vec{k})} \end{aligned} \tag{3.17}$$

where the linear interaction coefficient $L^{(\alpha',\alpha)}_{(\vec{k})}$ is given by

$$L^{(\alpha',\alpha)}_{(\vec{k})} = \left\langle \mathcal{L}_S(i\vec{k}) \vec{r}^{\,(\alpha')}_{(\vec{k})}, \vec{r}^{\,(\alpha)}_{(\vec{k})} \right\rangle. \tag{3.18}$$

In Eq. (3.18) $\mathcal{L}_S(i\vec{k})$ is the matrix symbol given by

$$\mathcal{L}_S(i\vec{k}) = (Ro)^{-1} \begin{pmatrix} -\dfrac{k_1 k_2}{|\vec{k}|^2} & -\dfrac{k_2^2 + k_3^2}{|\vec{k}|^2} & 0 & 0 \\ \dfrac{k_1^2 + k_3^2}{|\vec{k}|^2} & \dfrac{k_1 k_2}{|\vec{k}|^2} & 0 & 0 \\ -\dfrac{k_2 k_3}{|\vec{k}|^2} & \dfrac{k_1 k_3}{|\vec{k}|^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.19}$$

if $\vec{k} \neq 0$, and by

$$\mathcal{L}_S(i\vec{k}) = (Ro)^{-1} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.20}$$

when $\vec{k} = 0$. In similar fashion, the contribution from the diffusive operator $\mathcal{D}$ to the averaged equations is given by

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{s\mathcal{L}_F} \mathcal{D}\left(e^{-s\mathcal{L}_F} \bar{u}(\vec{x}, t)\right) \, ds =$$

$$= \sum_{\vec{k} \in Z^3} \sum_{\omega^{(\alpha')}_{(\vec{k})} = \omega^{(\alpha)}_{(\vec{k})}} D^{(\alpha', \alpha)}_{(\vec{k})} \sigma^{(\alpha')}_{(\vec{k})}(t) e^{i\vec{k}\cdot\vec{x}} \vec{r}^{(\alpha)}_{(\vec{k})} \tag{3.21}$$

where $D^{(\alpha', \alpha)}_{(\vec{k})}$ is given by

$$D^{(\alpha', \alpha)}_{(\vec{k})} = \langle \mathcal{D}(i\vec{k}) \vec{r}^{(\alpha')}_{(\vec{k})}, \vec{r}^{(\alpha)}_{(\vec{k})} \rangle \tag{3.22}$$

and the matrix symbol $\mathcal{D}(i\vec{k})$ is given by

$$\mathcal{D}(i\vec{k}) = - \begin{pmatrix} (Re)^{-1}|\vec{k}|^2 & 0 & 0 & 0 \\ 0 & (Re)^{-1}|\vec{k}|^2 & 0 & 0 \\ 0 & 0 & (Re)^{-1}|\vec{k}|^2 & 0 \\ 0 & 0 & 0 & (Re)^{-1}(Pr)^{-1}|\vec{k}|^2 \end{pmatrix} \tag{3.23}$$

Collecting the results from Eqs. (3.15), (3.17) and (3.21) back into the averaged equations in (2.33) gives

Concrete form of the averaged equations for the Fourier amplitudes $\sigma^{(\alpha)}_{(\vec{k})}(t)$

$$\frac{d\sigma^{(\alpha)}_{(\vec{k})}}{dt} + \sum_{\substack{\vec{k}'+\vec{k}''=\vec{k} \\ \omega^{(\alpha')}_{(\vec{k}')}+\omega^{(\alpha'')}_{(\vec{k}'')}=\omega^{(\alpha)}_{(\vec{k})}}} B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})} \sigma^{(\alpha')}_{(\vec{k}')} \sigma^{(\alpha'')}_{(\vec{k}'')} + \sum_{\omega^{(\alpha')}_{(\vec{k})}=\omega^{(\alpha)}_{(\vec{k})}} L^{(\alpha',\alpha)}_{(\vec{k})} \sigma^{(\alpha')}_{(\vec{k})} = \sum_{\omega^{(\alpha')}_{(\vec{k})}=\omega^{(\alpha)}_{(\vec{k})}} D^{(\alpha',\alpha)}_{(\vec{k})} \sigma^{(\alpha')}_{(\vec{k})} \tag{3.24}$$

where the quadratic interaction coefficient $B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})}$ is given in Eq. (3.12). the linear interaction coefficient $L^{(\alpha',\alpha)}_{(\vec{k})}$ is given in Eq. (3.18), and the diffusive interaction coefficient $D^{(\alpha',\alpha)}_{(\vec{k})}$ is given by Eq. (3.22).

3.3 Analysis of the Limiting Dynamics with Small Froude and Finite Rossby Numbers

The concrete form of the averaged equations in Eq. (3.24) displays the possible resonant interactions between fast and slow modes. We consider separately the dynamics of the slow and the fast modes.

(a) Slow Dynamics

Consider the limiting dynamics equations in eq. (3.24) for the slow mode with Fourier amplitude $\sigma^{(0)}_{(\vec{k})}$. In this case the equation for the three wave resonant interactions in Eq. (3.14) reduces to

$$\begin{aligned} \vec{k}' + \vec{k}'' &= \vec{k} \\ \omega^{(\alpha')}_{(\vec{k}')} + \omega^{(\alpha'')}_{(\vec{k}'')} &= 0 \, . \end{aligned} \tag{3.25}$$

The solutions of the resonant condition in Eq. (3.25) involve either two slow modes or two fast waves belonging to opposite wave families. However, in the case of two fast waves resonantly interacting with the slow mode, a direct calculation shows that the resonant interaction coefficient in Eq. (3.12) is always zero

$$B^{(1,-1,0)}_{(\vec{k}',\vec{k}'',\vec{k})} = B^{(-1,1,0)}_{(\vec{k}',\vec{k}'',\vec{k})} = 0 \tag{3.26}$$

for any resonant interaction in (3.25) involving two fast waves. Since the only possibility of resonant interactions with the fast waves was through the quadratic terms, we conclude that the averaged equations for the slow modes involve only resonant interactions with other slow modes and thus, the slow modes are free of interactions with the fast waves. In other words, the limiting dynamics of the slow modes is determined by the slow modes alone.

It follows that the reduced dynamics for the slow modes given in Eq. (3.24) in terms of the Fourier amplitudes is equivalent to the slow dynamics equations given in Eq. (2.37). To obtain a concrete representation of these equations we observe that in the low Fr and finite Ro regime, the operator $\mathcal{L}_F$ is given by Eq. (2.15) with its nullspace $N(\mathcal{L}_F)$ given by the hydrostatic balance and strong stratification conditions in Eq. (2.38). It is straightforward to verify that $N(\mathcal{L}_F)$ is also characterized by the conditions

$$div_H\, \vec{v}_H = 0, \quad w = 0, \quad \rho = \rho(x_3)\ . \tag{3.27}$$

and that the orthogonal projection P onto $N(\mathcal{L}_F)$ is given by

$$P\vec{u} = \begin{pmatrix} \vec{v}_H - \nabla_H\, \Delta_H^{-1}\, div_H\, \vec{v}_H \\ \\ 0 \\ \\ \frac{1}{(2\pi)^2} \int \rho(\vec{x}_H, x_3) d\vec{x}_H \end{pmatrix}. \tag{3.28}$$

With this concrete form of the nullspace $N(\mathcal{L}_F)$ and the orthogonal projection P, it follows that the slow dynamics equations in (2.37) take the form

Limiting slow dynamics for small Fr and finite Ro numbers

$$\begin{aligned} &\frac{\partial \vec{v}_H}{\partial t} + \vec{v}_H \cdot\, \nabla_H\, \vec{v}_H + (Ro)^{-1}\vec{v}_H^{\perp} + \nabla_H\, \phi = (Re)^{-1}\Delta_H \vec{v}_H + (Re)^{-1}\frac{\partial^2 \vec{v}_H}{\partial x_3^2} \\ &div_H\, \vec{v}_H = 0 \\ &w \equiv 0 \\ &\frac{\partial \rho}{\partial t} = (Re)^{-1}(Pr)^{-1}\frac{\partial^2 \rho}{\partial x_3^2} \end{aligned} \tag{3.29}$$

where $\vec{v}_H^{\perp} = (-v_2, v_1)$. In the slow dynamics equations the velocity and the density are decoupled. The fluid is strongly stratified with zero vertical velocity w and density ρ that only changes in the vertical direction. The different horizontal layers of fluid exchange momentum through diffusion in the vertical direction. The pressure term $\nabla_H\, \phi$ is needed to enforce the zero horizontal divergence constraint. This pressure term is given explicitly in terms of the horizontal velocity by

$$\nabla_H\, \phi = \nabla_H\, \left((Ro)^{-1}\Delta_H^{-1}\omega - \Delta_H^{-1}\, div_H\, (\vec{v}_H \cdot\, \nabla_H\, \vec{v}_H)\right) \tag{3.30}$$

where $\omega = \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}$ is the vertical component of the vorticity. For the special case with $Ro = +\infty$, the equations in (3.29) have been proposed earlier on heuristic grounds by Riley et al. (1981). Here they emerge as an automatic consequence of general systematic averaging principles with complete mathematical rigor.

Next we derive the vorticity-stream form of the limiting slow dynamics equations in (3.29). We start with the fact that any periodic horizontal vector field $\vec{v}_H(\vec{x}_H, x_3)$ with zero horizontal divergence, $div_H\, \vec{v}_H = 0$, has a unique decomposition in terms of a vertical shear flow $\vec{V}_H(x_3)$ and a stream function $\psi(\vec{x}_H, x_3)$

$$\vec{v}_H = \vec{V}_H + \nabla_H^{\perp}\psi \tag{3.31}$$

where $\nabla_H^{\perp}\psi = \left(-\dfrac{\partial\psi}{\partial x_2}, \dfrac{\partial\psi}{\partial x_1}\right)$. In this decomposition the vertical shear component $\vec{V}_H$ represents the horizontal average of the velocity field $\vec{v}_H$, whereas the component produced by the stream function ψ represents the component of the velocity field that has spatial dependence in the horizontal variables. In terms of the Fourier basis of eigenfunctions given by Eqs. (3.4), (3.5), and (3.6), it follows that the vertical shear component $\vec{V}_H$ is a superposition of the slow vortical modes with $\vec{k}_H = 0$ and $\vec{k} \neq 0$, and the remaining component $\nabla_H^{\perp}\psi$ of $\vec{v}_H$ is the superposition of the slow vortical modes with $\vec{k}_H \neq 0$. In addition, from the decomposition of $\vec{v}_H$ given in Eq. (3.31) it follows that the vertical component of the vorticity ω is related to the stream function ψ by

$$\omega = \Delta_H\psi \tag{3.32}$$

where Δ_H is the Laplacian in the horizontal variables. Therefore, computing the horizontal average and the vertical component of the vorticity from the horizontal momentum equation in Eq. (3.29) yields

Vorticity-stream form of the limiting slow dynamics equations for low Fr and finite Ro numbers

1. The horizontal velocity is $\vec{v}_H = \vec{V}_H + \nabla_H^{\perp}\psi$, where the vertical shear $\vec{V}_H$ satisfies

$$\frac{\partial}{\partial t}\vec{V}_H + (Ro)^{-1}\vec{V}_H^{\perp} = (Re)^{-1}\frac{\partial^2}{\partial x_3^2}\vec{V}_H \tag{3.33}$$

and the vertical vorticity ω and ψ satisfy the vorticity-stream equations

$$\begin{aligned}&\frac{\partial\omega}{\partial t} + \vec{V}_H \cdot \nabla_H\,\omega + J_H(\psi,\omega) = (Re)^{-1}\Delta_H\omega + (Re)^{-1}\frac{\partial^2\omega}{\partial x_3^2}\\ &\Delta_H\psi = \omega\end{aligned} \tag{3.34}$$

where $J_H(\psi, \omega) = \nabla_H^{\perp}\psi \cdot \nabla_H\, \omega$ is the Jacobian of ψ and ω in the horizontal variables.

2. The equations for the vertical velocity w and the density ρ are the same as in Eq. (3.29)

$$
\begin{aligned}
&w \equiv 0 \\
&\frac{\partial \rho}{\partial t} = (Re)^{-1}(Pr)^{-1}\frac{\partial^2 \rho}{\partial x_3^2}
\end{aligned}
\tag{3.35}
$$

¿From Eq. (3.33) it is clear that the dynamics of the vertical shear component $\vec{V}_H$ of $\vec{v}_H$ evolves independently of the stream function component $\nabla_H^{\perp}\psi$. In fact, the dynamics in Eq. (3.33) is given by a linear equation with constant coefficients and it is readily solved in terms of Fourier series. If initially the vertical shear $\vec{V}_{H0}(x_3)$ is given by

$$
\vec{V}_{H0}(x_3) = \sum_{k_3=-\infty}^{\infty} e^{ik_3x_3}\widehat{\vec{V}_{H0}}(k_3) \tag{3.36}
$$

then the vertical shear $\vec{V}_H(x_3, t)$ at later time is given by

$$
\vec{V}_H(x_3, t) = \sum_{k_3=-\infty}^{\infty} e^{-(Re)^{-1}k_3^2 t} e^{ik_3x_3} R(t)\widehat{\vec{V}_{H0}}(k_3) \tag{3.37}
$$

where $R(t)$ is the rotation matrix

$$
R(t) = \begin{pmatrix} \cos(R_0^{-1}t) & \sin(R_0^{-1}t) \\ -\sin(R_0^{-1}t) & \cos(R_0^{-1}t) \end{pmatrix} \tag{3.38}
$$

The formulas in Eqs. (3.37) and (3.38) show that $\vec{V}_H(x_3, t)$ represents a time decaying vertical shear flow that rotates with uniform angular velocity of order $O((Ro)^{-1})$. On the other hand, the vertical shear flow $\vec{V}_H$ influences the dynamics of the stream function component $\nabla_H^{\perp}\psi$ and provides an important mechanism for energy transfers among different horizontal layers of fluid. Indeed, it is clear from Eq. (3.34) that the vertical vorticity ω is advected by the vertical shear $\vec{V}_H$. Therefore the variations of $\vec{V}_H$ in the vertical direction will in turn induce corresponding higher vertical gradients for ω which will be dissipated through vertical diffusion. Elementary solutions of the vorticity-stream form of the slow dynamics equations in (3.34) will be discussed in the last section of the paper.

(b) Fast Dynamics

We consider the limiting dynamics equations in (3.24) for the fast modes $\sigma_{(\vec{k})}^{(\pm 1)}$ From the linear analysis previously done, we know that in the case of $\vec{k} \neq 0$ these fast modes are associated to the eigenvectors $\vec{r}_{(\vec{k})}^{(\pm 1)}$ in Eq. (3.4), and in the case of $\vec{k} = 0$ they are associated to $\vec{r}_{(\vec{k})}^{(\pm 1)}$ in Eq. (3.6). We start by considering the three wave resonance interaction condition in (3.14) for the case $\vec{k} \neq 0$

$$\begin{gathered} \vec{k}' + \vec{k}'' = \vec{k} \\ \omega_{(\vec{k}')}^{(\alpha')} + \omega_{(\vec{k}'')}^{(\alpha'')} = \omega_{(\vec{k})}^{(\pm 1)} = \pm \frac{|\vec{k}_H|}{|\vec{k}|} . \end{gathered} \tag{3.39}$$

In this case the three wave resonant interaction condition in Eq. (3.39) can be satisfied by two fast modes and one slow mode (Slow-Fast-Fast), or by three fast modes (Fast-Fast-Fast). Next we consider both possibilities.

Slow-Fast-Fast Resonant Interactions

This type of interaction occurs when a slow mode resonates with two fast modes of a given family. In this case the three wave resonance condition in Eq. (3.39) reduces to

$$\begin{gathered} \vec{k}' + \vec{k}'' = \vec{k} \\ \frac{|\vec{k}'_H|}{|\vec{k}'|} = \frac{|\vec{k}_H|}{|\vec{k}|} \end{gathered} \tag{3.40}$$

where we assumed that $\vec{k}'$ is associated with the fast mode and $\vec{k}''$ is associated to the slow mode. The resonance condition in Eq. (3.40) is satisfied provided the two wave numbers $\vec{k}$ and $\vec{k}'$ of the fast waves lie in the same vertical cone (Lelong and Riley, 1991). To compute the quadratic interaction coefficient $B_{(\vec{k}',\vec{k}'',\vec{k})}^{(\pm 1,0,\pm 1)}$ we utilize Eq. (3.12) in combination with the formulas for the right eigenvectors given earlier in eqs. (3.4), (3.5), and (3.6). When $\vec{k}''_H$ satisfies $\vec{k}''_H \neq 0$ the interaction coefficient is given by

$$B_{(\vec{k}',\vec{k}'',\vec{k})}^{(\pm 1,0,\pm 1)} = -\frac{1}{2} \frac{(\vec{k}_H^{\perp} \cdot \vec{k}'_H)}{|\vec{k}'_H| |\vec{k}''_H| |\vec{k}_H| |\vec{k}'| |\vec{k}|} \left[(\vec{k}'_H \cdot \vec{k}_H) k'_3 k_3 + |\vec{k}'_H|^2 |\vec{k}_H|^2 \right] . \tag{3.41}$$

On the other hand, when $\vec{k}''_H = 0$ there are three possible slow modes in Eq. (3.5) and these include the Slow-Fast-Fast interactions of vertical shear $\vec{V}_H$, at a wave number $(0, 0, k'_3)$, as well as perturbed vertical density variations in hydrostatic balance. In this case the values of the corresponding quadratic interaction coefficients are

$$B^{(\pm1,0,\pm1)}_{(\vec{k}',\vec{k}'',\vec{k})} = \frac{1}{2}\frac{\left|\vec{k}_H\right|}{\left|\vec{k}\right|}k_3''.$$

$$B^{(\pm1,1,\pm1)}_{(\vec{k}',\vec{k}'',\vec{k})} = \frac{1}{2\sqrt{2}}(-k_1 + ik_2) \tag{3.42}$$

$$B^{(\pm1,-1,\pm1)}_{(\vec{k}',\vec{k}'',\vec{k})} = \frac{1}{2\sqrt{2}}(k_1 + ik_2).$$

The last two coefficients in (3.42) represent the interactions produced from the vertical shear. Finally, when $\vec{k}'' = 0$, there are two possible slow modes in Eq. (3.6), $\vec{r}^{(0)}_{(0)}$ and $\tilde{\vec{r}}^{(0)}_{(0)}$. In this case the quadratic interaction coefficients reduce to

$$\begin{aligned} B^{(\pm1,0,\pm1)}_{(\vec{k},0,\vec{k})} &= \frac{ik_1}{2} \\ \tilde{B}^{(\pm1,0,\pm1)}_{(\vec{k},0,\vec{k})} &= \frac{ik_2}{2} \end{aligned} \tag{3.43}$$

It is clear that in general the coefficients in (3.41), (3.42) and (3.43) are nonzero. We remark that the fast mean modes in Eq. (3.6) do not participate in Slow-Fast-Fast resonant interactions.

Fast-Fast-Fast Resonant Interactions

The resonant wave interaction of three waves may involve either three waves of the same family, or two waves from one family and one wave from the opposite family. In the case of three waves from the same family, the three wave resonance condition in Eq. (3.14) becomes

$$\begin{aligned} &\vec{k}' + \vec{k}'' = \vec{k} \\ &\frac{\left|\vec{k}'_H\right|}{\left|\vec{k}'\right|} + \frac{\left|\vec{k}''_H\right|}{\left|\vec{k}''\right|} = \frac{\left|\vec{k}_H\right|}{\left|\vec{k}\right|} \end{aligned} \tag{3.44}$$

and the quadratic interaction coefficient in Eq. (3.12) is given by

$$B^{(\pm1,\pm1,\pm1)}_{(\vec{k}',\vec{k}'',\vec{k})} = -\frac{1}{4\sqrt{2}}\left(\frac{(\vec{k}'_H\cdot\vec{k}''_H)k'_3}{\left|\vec{k}'_H\right|\left|\vec{k}'\right|} - \frac{\left|\vec{k}'_H\right|}{\left|\vec{k}'\right|}k''_3\right)\left[\frac{(\vec{k}''_H\cdot\vec{k}_H)k''_3k_3}{\left|\vec{k}''_H\right|\left|\vec{k}_H\right|\left|\vec{k}''\right|\left|\vec{k}\right|} + \frac{\left|\vec{k}''_H\right|\left|\vec{k}_H\right|}{\left|\vec{k}''\right|\left|\vec{k}\right|} + 1\right]$$
$$+\left(\frac{(\vec{k}''_H\cdot\vec{k}'_H)k''_3}{\left|\vec{k}''_H\right|\left|\vec{k}''\right|} - \frac{\left|\vec{k}''_H\right|}{\left|\vec{k}''\right|}k'_3\right)\left[\frac{(\vec{k}'_H\cdot\vec{k}_H)k'_3k_3}{\left|\vec{k}'_H\right|\left|\vec{k}_H\right|\left|\vec{k}'\right|\left|\vec{k}\right|} + \frac{\left|\vec{k}'_H\right|\left|\vec{k}_H\right|}{\left|\vec{k}'\right|\left|\vec{k}\right|} + 1\right]. \tag{3.45}$$

Similarly, in the case of a three wave interaction involving two fast waves from one family and another fast wave from the opposite family, the resonance condition now becomes

$$\vec{k}' + \vec{k}'' = \vec{k}$$
$$\frac{\left|\vec{k}'_H\right|}{\left|\vec{k}'\right|} - \frac{\left|\vec{k}''_H\right|}{\left|\vec{k}''\right|} = \frac{\left|\vec{k}_H\right|}{\left|\vec{k}\right|} \tag{3.46}$$

and the corresponding quadratic interaction coefficient is

$$B^{(\pm1,\mp1,\pm1)}_{(\vec{k}',\vec{k}'',\vec{k})} = \frac{1}{4\sqrt{2}}\left(\frac{(\vec{k}'_H\cdot\vec{k}''_H)k'_3}{\left|\vec{k}'_H\right|\left|\vec{k}'\right|} - \frac{\left|\vec{k}'_H\right|}{\left|\vec{k}'\right|}k''_3\right)\left[\frac{(\vec{k}''_H\cdot\vec{k}_H)k''_3k_3}{\left|\vec{k}''_H\right|\left|\vec{k}_H\right|\left|\vec{k}''\right|\left|\vec{k}\right|} + \frac{\left|\vec{k}''_H\right|\left|\vec{k}_H\right|}{\left|\vec{k}''\right|\left|\vec{k}\right|} - 1\right]$$
$$+\left(\frac{(\vec{k}''_H\cdot\vec{k}'_H)k''_3}{\left|\vec{k}''_H\right|\left|\vec{k}''\right|} - \frac{\left|\vec{k}''_H\right|}{\left|\vec{k}''\right|}k'_3\right)\left[\frac{(\vec{k}'_H\cdot\vec{k}_H)k'_3k_3}{\left|\vec{k}'_H\right|\left|\vec{k}_H\right|\left|\vec{k}'\right|\left|\vec{k}\right|} + \frac{\left|\vec{k}'_H\right|\left|\vec{k}_H\right|}{\left|\vec{k}'\right|\left|\vec{k}\right|} + 1\right]. \tag{3.47}$$

The question now is whether or not there are any solutions of the three wave resonance equations in Eq. (3.44) and (3.46). The following example shows that there are non-trivial solutions of the resonance condition in Eq. (3.44): let $\vec{k}' = (0,3,6)$, $\vec{k}'' = (0,5,-10)$, and $\vec{k} = \vec{k}' + \vec{k}'' = (0,8,-4)$. In this case the corresponding frequencies are $\omega^{(1)}_{(\vec{k}')} = 1/\sqrt{5}$, $\omega^{(1)}_{(\vec{k}'')} = 1/\sqrt{5}$, and $\omega^{(1)}_{(\vec{k})} = 2/\sqrt{5}$, hence the resonance condition in Eq. (3.44) is satisfied. This example is easily modified to produce solutions of the resonance condition in eq. (3.46). This is simply done by taking $\vec{k}' = (0,8,-4)$, $\vec{k}'' = (0,-5,10)$, and $\vec{k} = \vec{k}' + \vec{k}'' = (0,3,6)$. In this case the corresponding frequencies are $\omega^{(1)}_{(\vec{k}')} = 2/\sqrt{5}$, $\omega^{(1)}_{(\vec{k}'')} = 1/\sqrt{5}$, and $\omega^{(1)}_{(\vec{k})} = 1/\sqrt{5}$ and the resonance condition in (3.46) is satisfied. In fact, the solutions of the resonance equation in (3.44) are not all that sparse. For example, if it is not hard to compute numerically that there are 224 resonant triad solutions of Eq. (3.44) with $\vec{k}'$ and $\vec{k}''$ lying in the cube with $|k_i| \leq 30$ for $i = 1,2,3$. A short list of some of these resonant triad solutions of Eq. (3.44) is given in Table 1. Such three-wave resonant interactions of internal gravity waves have been studied extensively (McComas and Bretherton, 1977; Muller, et al., 1986). The explicit examples mentioned above involve parametric resonances.

Conservation in Time of the Slow-Fast Energy Ratio

Finally, another important feature of the limiting dynamics equations for low Froude and finite Rossby numbers is the constancy of the energy ratio between the vortical modes and gravity modes when there are no diffusive effects. To see why this is so, consider a solution $\vec{u}^{\epsilon}$ of the rotating Boussinesq equations, and for simplicity assume that there are no diffusion effects. According to Eq. (2.34) the solution $\vec{u}^{\epsilon}$ is of the form

$$\vec{u}^{\epsilon}(\vec{x},t) = e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}(\vec{x},t) + o(1) \tag{3.48}$$

where $\bar{u}$ is the solution of the limiting dynamics equations. Next decompose the solution $\bar{u}$ into its slow (vortical) component $\bar{u}^S$ and fast (gravity) component $\bar{u}^F$

$$\bar{u} = \bar{u}^S + \bar{u}^F \ . \tag{3.49}$$

If we substitute Eq. (3.49) into Eq. (3.50) we get

$$\vec{u}^{\epsilon}(\vec{x},t) = \bar{u}^S(\vec{x},t) + e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}^F(\vec{x},t) + o(1) \ . \tag{3.50}$$

Now let $||\cdot||^2$ denote the sum of the potential and kinetic energies. Since the operator $\mathcal{L}_F$ is skew-hermitian in the space of divergence free velocity fields with the energy norm defined above, then the slow component $\bar{u}^S$ and the fast component $e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}^F$ are orthogonal, and the operator $e^{-\epsilon^{-1}t\mathcal{L}_F}$ conserves energy. Therefore computing the total energy of $\vec{u}^{\epsilon}$ in Eq. (3.50) yields

$$||\vec{u}^{\epsilon}||^2 = \left|\left|\bar{u}^S\right|\right|^2 + \left|\left|e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}^F\right|\right|^2 + o(1) = \left|\left|\bar{u}^S\right|\right|^2 + \left|\left|\bar{u}^F\right|\right|^2 + o(1) \tag{3.51}$$

In the absence of diffusion Eq. (2.27) shows that the rotating Boussinesq equations conserve total energy, so that $||\vec{u}^{\epsilon}|| = ||\vec{u}(0)||$, where $\vec{u}(0)$ is the initial data. Moreover, the limiting slow dynamics equations for $\bar{u}^S$ in (3.29) also conserve energy in the absence of diffusion

$$\left|\left|\bar{u}^S(t)\right|\right| = \left|\left|\bar{u}^S(0)\right|\right| \ . \tag{3.52}$$

Therefore, when ϵ goes to zero the energy identity in eq. (3.51) reduces to

$$||\vec{u}(0)||^2 = \left|\left|\bar{u}^S\right|\right|^2 + \left|\left|\bar{u}^F\right|\right|^2 = \left|\left|\bar{u}^S(0)\right|\right|^2 + \left|\left|\bar{u}^F\right|\right|^2 \tag{3.53}$$

and from this we conclude that the energy of the slow component $\bar{u}^S$ and the fast component $\bar{u}^F$ are conserved. This demonstrates that the energy ratio of the vortical modes and the gravity waves is conserved in time. In particular, this provides a theoretical explanation for the constancy of this ratio observed in numerical simulations of freely evolving turbulence in stably stratified fluids at low Froude numbers by Metais and Herring (1989) on advective time scales.

4. LOW FROUDE NUMBER AND LOW ROSSBY NUMBER LIMITING DYNAMICS

Next we study the limiting dynamics equations (2.33) for the rotating Boussinesq equations when both the Froude number Fr and the Rossby number Ro are small. In this case the scaling is given by Eq. (2.18), where $Fr = \epsilon$, and the Rossby and Froude number are assumed to be in constant ratio $F = Fr/Ro$. We know that in both the asymptotic regime of low Fr and low Ro numbers, and in the asymptotic regime of low Fr and finite Ro numbers, the Boussinesq equations retain the same structure given by Eq. (2.26). In addition, the corresponding skew-hermitian linear operators $\mathcal{L}_F$ given by Eq. (2.15) Eq. (2.20) have orthogonal basis of eigenfunctions so that any incompressible vector field $\bar{u}(\vec{x}, t)$ has the expansion of the form given in Eq. (3.8). Therefore, the derivation of the derivation of the concrete form of the averaged equations given in (3.24) for the low Fr and finite Ro regime also applies in the present regime of low Fr and low Ro numbers. First we discuss the limiting slow dynamics and then consider the concrete form of the limiting dynamics equations for the fast waves.

4.1 Limiting Slow Dynamics for the Boussinesq Equations with Slanted Rotation and the Quasi-geostrophic Approximation

Next we consider the limiting slow dynamics equations for the Boussinesq equations with slanted rotation. It was already mentioned in Sec. 2 that the slow dynamics equations are determined by the slow modes alone and that there are no gravity waves, that is, the quadratic interaction terms involving two gravity waves of opposite families and one slow vortical mode are zero. Therefore, the limiting dynamics for the slow modes satisfies the geostrophic-hydrostatic balance and strong stratification condition in Eq. (2.39)

$$\begin{aligned} F\vec{\eta} \times \vec{v} + \rho\vec{e}_3 + \nabla\phi = 0 \\ w = 0 \end{aligned} \tag{4.1}$$

where $\nabla\phi = \nabla\Delta^{-1}\left(F\vec{\eta}\cdot\vec{\omega} - \dfrac{\partial\rho}{\partial x_3}\right)$. In addition, the slow modes satisfy the limiting form of the potential vorticity equation in (2.21)

$$\frac{D}{Dt}\left[\omega - F(\nabla\rho\cdot\vec{\eta})\right] = (Re)^{-1}\Delta\left[\omega \ - (Pr)^{-1}F(\nabla\rho\cdot\vec{\eta})\right] \tag{4.2}$$

where $\omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$ is the vertical component of the vorticity. Next we show that Eqs. (4.1) and (4.2) yield the quasigeostrophic equations with slanted rotation. Since the incompressible velocity field $\vec{v}$ in Eq. (4.1) is strongly stratified, $w = 0$, then the horizontal component of the velocity is $\vec{v}_H$ is divergence free

$$div_H \; \vec{v}_H = 0. \tag{4.3}$$

To study the geostrophic-hydrostatic balance condition in (4.1) we consider two cases depending on whether or not $\vec{\eta}$ is orthogonal to the direction of gravity $\vec{e}_3$. If $\vec{\eta}$ is not orthogonal to $\vec{e}_3$ then the geostrophic-hydrostatic balance condition in (4.1) becomes componentwise

$$\begin{aligned} &-F\eta_3 v_2 + \frac{\partial \phi}{\partial x_1} = 0 \\ &F\eta_3 v_1 + \frac{\partial \phi}{\partial x_2} = 0 \\ &F(\eta_1 v_2 - \eta_2 v_1) + \frac{\partial \phi}{\partial x_3} + \rho = 0 \end{aligned} \tag{4.4}$$

The first two equations in Eq. (4.4) yield the standard geostrophic balance between Coriolis and pressure forces. The third equation is a modification of the standard hydrostatic balance approximation where the effects of Coriolis forces due to the slanted axis of rotation need to be added in the vertical direction, with the resulting effect that we now have a coupled geostrophic-hydrostatic balance condition in the vertical direction. The horizontal components equations in (4.4) yield the stream function $\psi = F^{-1}\eta_3^{-1}\phi$, so that

$$\vec{v}_H = \nabla_H^{\perp}\psi \tag{4.5}$$

and the incompresssibility condition for the horizontal component $\vec{v}_H$ is automatically satisfied. Introducing Eq. (4.5) into the third equation in (4.4) and solving for the density yields

$$\rho = -F\frac{\partial \psi}{\partial \eta} \tag{4.6}$$

where $\frac{\partial \psi}{\partial \eta} = \nabla\psi \cdot \eta$ is the directional derivative in the direction of η. From here it directly follows that

$$\nabla\rho \cdot \eta = \frac{\partial \rho}{\partial \eta} = -F\frac{\partial^2 \psi}{\partial \eta^2}. \tag{4.7}$$

Next we consider the case where the axis of rotation $\vec{\eta}$ is orthogonal to the direction of gravity, $\eta_3 = \vec{\eta} \cdot \vec{e}_3 = 0$ as arises in a tangent approximation at the equator. The geostrophic-hydrostatic balance conditions become

$$\begin{aligned} &\frac{\partial \phi}{\partial x_1} = 0 \\ &\frac{\partial \phi}{\partial x_2} = 0 \\ &F(\eta_1 v_2 - \eta_2 v_1) + \frac{\partial \phi}{\partial x_3} + \rho = 0. \end{aligned} \tag{4.8}$$

In this case the horizontal component of the velocity $\vec{v}_H$ is not in geostrophic balance with the pressure. In fact, the first two equations in (4.8) imply that the pressure is a function of the vertical variable x_3 alone. However, the incompressibility condition in Eq. (4.5) still guarantees the existence of a stream function ψ with $\vec{v}_H = \nabla_H^{\perp}\psi$. Therefore solving for ρ in the third equation in (4.8) gives

$$\rho = -F\frac{\partial \psi}{\partial \eta} + \frac{d\phi}{dx_3} \tag{4.9}$$

where $\vec{\eta} = (\eta_1, \eta_2, 0)$. Therefore, if we differentiate again in the direction of $\vec{\eta}$, this yields only the contribution from ψ

$$\nabla \rho \cdot \eta = \frac{\partial \rho}{\partial \eta} = -F\frac{\partial^2 \psi}{\partial \eta^2} \tag{4.10}$$

which coincides with formula for $\nabla\rho \cdot \vec{\eta}$ previously derived in Eq. (4.7) for the case where $\vec{\eta}$ and $\vec{e}_3$ are not orthogonal.

Finally, we recast the potential vorticity equation in terms of the stream function ψ. Since $\vec{v}_H = \nabla_H^{\perp}\psi$ then the vertical component of the vorticity is $\omega = \Delta_H\psi$, where Δ_H is the Laplacian in the horizontal variables. Substituting this value of ω and the value of $\nabla\rho \cdot \vec{\eta}$ back into the potential vorticity equation yields

The Quasi-Geostrophic Equations for Slanted Rotation

$$\begin{aligned} &\vec{v}_H = \nabla_H^{\perp}\psi \\ &\frac{D^H}{Dt}\left(\Delta_H\psi + F^2\frac{d^2\psi}{d\eta^2}\right) = (Re)^{-1}\Delta\left(\Delta_H\psi + (Pr)^{-1}F^2\frac{d^2\psi}{d\eta^2}\right) \end{aligned} \tag{4.11}$$

where $\frac{D^H}{Dt} = \frac{\partial}{\partial t} + \vec{v}_H(\vec{x}, t) \cdot \nabla_H$. In particular, it is interesting to observe that when $\eta_3 = 0$, Eq. (4.11) gives a horizontal anisotropic correction $F^2\frac{d^2\psi}{d\eta^2}$ to the vertical vorticity $\omega = \Delta_H\psi$.

4.2 The Quasigeostrophic Equations for Slanted Rotation and Unbalanced Initial Data

Consider the solution $\vec{u}(\vec{x},t)$ of the limiting dynamics equations in the limit of small Fr and small Ro numbers. If the initial data $\vec{u}_0(\vec{x})$ has no fast gravity wave components, i.e. the initial data is in geostrophic-hydrostatic balance, then the solution will remain in balance to leading order and its dynamics is described by the quasigeostrophic equation for slanted rotation in (4.11). Moreover, if $\vec{u}^{\epsilon}(\vec{x},t)$ is the solution of the Boussinesq equations with slanted rotation in Eqs. (2.19) and (2.20) that satisfies the same initial condition $\vec{u}_0(\vec{x})$, then the classical techniques from singular limits for hyperbolic equations (Klainerman and Majda, 1981; Majda, 1984) can be utilized to show that $\vec{u}^{\epsilon}(\vec{x},t)$ converges in the strong (classical) sense to the solution $\vec{u}(\vec{x},t)$ of the quasigeostrophic equations. On the other hand, we know that the solution $\vec{u}^{\epsilon}(\vec{x},t)$ and the solution $\bar{u}(\vec{x},t)$ of the limiting dynamics equations are related by Eq. (2.34)

$$\vec{u}^{\epsilon}(\vec{x},t) = e^{-\epsilon^{-1}t\mathcal{L}_F}\bar{u}(\vec{x},t) + o(1) \; . \tag{4.12}$$

As a consequence, if the initial data has fast (gravity) wave components, then the solution $\vec{u}^{\epsilon}(\vec{x},t)$ will have a persistent highly oscillatory component and it will be impossible to produce strong convergence for the approximations. However, suppose that we decompose $\bar{u}$ as

$$\bar{u} = \bar{u}^S + \bar{u}^F \tag{4.13}$$

where $\bar{u}^S$ is the slow (vortical) component of $\bar{u}$ and $\bar{u}^F$ is the fast (gravity) component of $\bar{u}$. Then from Eq. (4.12) it follows that

$$\vec{u}^{\epsilon} = \bar{u}^S + \vec{U}^{\epsilon} + o(1) \tag{4.14}$$

where $\vec{U}^{\epsilon}$ has the fast oscillations. We claim that the solution $\vec{u}^{\epsilon}$ converges to the slow component $\bar{u}^S$ in the weak sense. By this we mean that there is convergence after we filter the solution with a smooth function $\Psi(\vec{x},t)$ of compact support in space-time, in the sense that

$$\lim_{\epsilon\to 0}\int_0^\infty \int_{T^3} \vec{u}^{\epsilon}(\vec{x},t)\Psi(\vec{x},t)\,dx\,dt = \int_0^\infty \int_{T^3} \bar{u}^S(\vec{x},t)\Psi(\vec{x},t)\,dx\,dt \; , \tag{4.15}$$

Now the question is what are the equations satisfied by the weak limit. The remarkable answer is that $\bar{u}^S$ satisfies in a similar filtered weak sense the quasigeostrophic equations for slanted rotation in (4.11). The argument follows along the lines given previously in Embid and Majda (1996) for the case with a vertical axis of rotation. In the decomposition $\bar{u} = \bar{u}^S + \bar{u}^F$, the slow component $\bar{u}^S$ satisfies the geostrophic-hydrostatic balance condition $\mathcal{L}_F(\bar{u}^S) = 0$ by definition. What remains is to check that $\bar{u}^S$ satisfies the potential vorticity equation in (4.2)

$$\frac{D\bar{q}^S}{Dt} = 0 \tag{4.16}$$

where $\bar{q}^S = \omega_3 - F(\nabla\rho \cdot \vec{\eta})$; here for simplicity we have omitted the diffusion terms in Eq. (4.2). To prove the validity of Eq. (4.16) we utilize Ertel's theorem on conservation of potential vorticity in Eq. (2.21)

$$\frac{Dq^{\epsilon}}{Dt} = 0 \tag{4.17}$$

where the potential vorticity q^{ϵ} is given by

$$q^{\epsilon} = \omega_3 - F(\nabla\rho \cdot \vec{\eta}) - \epsilon\,\vec{\omega} \cdot \nabla\rho\ . \tag{4.18}$$

If we filter the solution and multiply the potential vorticity equation in (4.18) by a space-time test function Ψ, filter the solution and integrate by parts, we get

$$\int_0^{\infty}\int_{T^3} \Psi_t q^{\epsilon} + \vec{v}^{\epsilon} q^{\epsilon} \cdot \nabla\Psi \, dx\, dt = 0 \tag{4.19}$$

To get weak convergence in Eq. (4.19) we utilize the remarkable fact that *to leading order in ϵ there are no fast wave contributions in the potential vorticity q^{ϵ}*, i.e.,

$$q^{\epsilon} = \bar{q}^S + o(1) \tag{4.20}$$

where $\bar{q}^S = \bar{\omega}_3 - F(\nabla\bar{\rho} \cdot \vec{\eta})$. This fact is verified in the appendix. Since $\bar{u}^S$ satisfies the geostrophic-hydrostatic balance conditions, then the velocity field $\vec{v}^{\epsilon}$ has the form

$$\vec{v}^{\epsilon} = \nabla_H^{\perp}\psi + \vec{V}^{\epsilon} + o(1)$$

where $\vec{V}^{\epsilon}$ contains the fast oscillations. Taking the limit in Eq. (4.19) yields

$$0 = \lim_{\epsilon \to 0}\int_0^{\infty}\int_{T^3} \Psi_t q^{\epsilon} + \vec{v}^{\epsilon} q^{\epsilon} \cdot \nabla\Psi \, dx\, dt = \int_0^{\infty}\int_{T^3} \Psi_t \bar{q}^S + \bar{q}^S \nabla_H^{\perp}\psi \cdot \nabla\Psi \, dx\, dt \tag{4.21}$$

which is the "averaged" weak form of the potential vorticity equation in (4.17). The results given by Eq. (4.15) and Eq. (4.21) can be interpreted as follows. Even tough the solution $\vec{u}^{\epsilon}$ in Eq. (4.14) has fast oscillations, only the slow component $\bar{u}^S$ persists through space-time filtering for $\epsilon \ll 1$. Moreover, Eq. (4.21) shows that $\bar{u}^S$ solves the quasigeostrophic equations. In this sense, there is a nonlinear Rossby adjustment process for $\epsilon \ll 1$ through the space-time filtering process described earlier.

4.3 Low Fr and Low Ro Limiting Dynamics of the Fast Waves for the Rotating Boussinesq Equations

As remarked earlier, the derivation of the concrete form of the averaged equations carried out in Sec. 3 for the regime of low Fr and finite Ro numbers also applies to the Boussinesq equations in the regime of low Fr and low Ro numbers, provided that we have the eigenfunctions of the corresponding operator $\mathcal{L}_F$. For this reason we start with the spectral analysis of the operator $\mathcal{L}_F$ and then discuss the resulting limiting dynamics equations for the fast waves. For computational simplicity we will only consider the case where the axis of rotation $\vec{\eta}$ is vertical, $\vec{\eta} = \vec{e}_3$.

Spectral Analysis of $\mathcal{L}_F$

For simplicity we assume periodic boundary conditions for the spatial domain and look for the Fourier eigenfunctions of the operator $\mathcal{L}_F$. These periodic eigenfunctions $\vec{u}$ are of the form

$$\vec{u} = \exp(i\vec{k}\cdot\vec{x} - i\omega(\vec{k})t)\vec{r} \tag{4.22}$$

where $\vec{u}(\vec{x},t)$ must satisfy the incompressibility condition. There are three cases to consider depending on whether $\vec{k}_H \neq 0$, or $\vec{k}_H = 0$ but $\vec{k} \neq 0$. or else $\vec{k} = 0$. In the first case where $\vec{k}_H \neq 0$ the eigenfrequencies $\omega(\vec{k})$ associated to the wave number $\vec{k}$ are

$$\begin{aligned}
\omega^{(-1)}_{(\vec{k})} &= -\omega(\vec{k}) = -\frac{(|\vec{k}_H|^2 + F^2 k_3^2)^{1/2}}{|\vec{k}|} \\
\omega^{(0)}_{(\vec{k})} &= 0, \text{ (double)} \\
\omega^{(1)}_{(\vec{k})} &= \omega(\vec{k}) = \frac{(|\vec{k}_H|^2 + F^2 k_3^2)^{1/2}}{|\vec{k}|}
\end{aligned} \tag{4.23}$$

and the corresponding right eigenvectors are given by

$$\begin{matrix} \vec{r}^{\,(-1)}_{(\vec{k})} & \vec{r}^{\,(1)}_{(\vec{k})} & \vec{r}^{\,(0)}_{(\vec{k})} \\
\begin{pmatrix} \dfrac{-Fk_2k_3 - i\omega(\vec{k})k_1k_3}{\sqrt{2}\omega(\vec{k})|\vec{k}_H||\vec{k}|} \\ \dfrac{Fk_1k_3 - i\omega(\vec{k})k_2k_3}{\sqrt{2}\omega(\vec{k})|\vec{k}_H||\vec{k}|} \\ i\dfrac{|\vec{k}_H|}{\sqrt{2}|\vec{k}|} \\ \dfrac{|\vec{k}_H|}{\sqrt{2}\omega(\vec{k})|\vec{k}|} \end{pmatrix} &
\begin{pmatrix} \dfrac{-Fk_2k_3 + i\omega(\vec{k})k_1k_3}{\sqrt{2}\omega(\vec{k})|\vec{k}_H||\vec{k}|} \\ \dfrac{Fk_1k_3 + i\omega(\vec{k})k_2k_3}{\sqrt{2}\omega(\vec{k})|\vec{k}_H||\vec{k}|} \\ -i\dfrac{|\vec{k}_H|}{\sqrt{2}|\vec{k}|} \\ \dfrac{|\vec{k}_H|}{\sqrt{2}\omega(\vec{k})|\vec{k}|} \end{pmatrix} &
\begin{pmatrix} -i\dfrac{k_2}{\omega(\vec{k})|\vec{k}|} \\ i\dfrac{k_1}{\omega(\vec{k})|\vec{k}|} \\ 0 \\ -i\dfrac{Fk_3}{\omega(\vec{k})|\vec{k}|} \end{pmatrix}
\end{matrix} \tag{4.24}$$

where the fourth eigenvector has been discarded because the corresponding eigensolution does not yield an incompressible velocity field. The eigenfunctions associated with $\vec{r}^{\,(\pm 1)}_{(\vec{k})}$ represent fast gravity waves, and the one associated with $\vec{r}^{\,(0)}_{(\vec{k})}$ represents a slow vortical mode. Notice that when the ratio $F = Fr/Ro$ is set to zero then Eqs. (4.23) and (4.24) reduce to the eigenfrequencies and the eigenvectors in Eqs. (3.2) and (3.4) for the low Fr and finite Ro regime.

In the second case when $\vec{k}_H = 0$ but $\vec{k} \neq 0$ the eigenfrequencies in Eq. (4.23) reduce to $\omega^{(\pm 1)}_{(\vec{k})} = \pm F$, and $\omega^{(0)}_{(\vec{k})} = 0$, and the corresponding eigenvectors are

$$
\begin{matrix} \vec{r}^{\,(-1)}_{(\vec{k})} & \vec{r}^{\,(0)}_{(\vec{k})} & \vec{r}^{\,(1)}_{(\vec{k})} \\
\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \end{matrix} . \tag{4.25}
$$

In this case the eigenvectors $\vec{r}^{\,(\pm 1)}_{(\vec{k})}$ correspond to fast rotation modes with the inertial rotation frequency, $\epsilon^{-1}F$, whereas the remaining eigenvector $\vec{r}^{\,(0)}_{(\vec{k})}$ yields a slow mode. In the final case of mean flows with $\vec{k} = 0$ there are four admissible eigenfunctions, two with eigenfrequency $\omega^{(\pm 1)}_{(0)} = \pm 1$, and two others with eigenfrequency $\tilde{\omega}^{(\pm 1)}_{(0)} = \pm F$. The corresponding eigenfunctions are

$$
\begin{matrix} \vec{r}^{\,(-1)}_{(0)} & \vec{r}^{\,(1)}_{(0)} & \tilde{\vec{r}}^{(-1)}_{(0)} & \tilde{\vec{r}}^{(1)}_{(0)} \\
\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \end{matrix} \tag{4.26}
$$

with $\vec{r}^{\,(\pm 1)}_{(0)}$ corresponding to fast gravity waves and $\tilde{\vec{r}}^{(\pm 1)}_{(0)}$ corresponding to fast rotation waves.

We remark that all the eigenvectors above have been normalized as to give orthonormal basis, and the corresponding Fourier eigenfunctions are an orthogonal family. In addition, the eigenvectors satisfy the symmetry condition in Eq. (3.7), so that the amplitude eigenfunction expansion in Eq. (3.8) yields a real valued solution if the amplitudes $\sigma^{(\alpha)}_{(\vec{k})}$ satisfy the symmetry condition in Eq. (3.9).

With the spectral analysis of the operator $\mathcal{L}_F$ concluded, we return to the analysis of the averaged equations in (3.24) for the Fourier amplitude $\sigma^{(\pm 1)}_{(\vec{k})}$ associated with one of the fast modes of the system. The three wave resonant condition for the existence of quadratic interactions in the limiting dynamics equations in (3.24) is given by Eq. (3.14), and allows for both Slow-Fast-Fast and Fast-Fast-Fast resonant interactions in addition to the slow interaction described earlier, which lead to the quasigeostrophic equation in (4.11) with $\vec{\eta} = (0, 0, 1)$.

Slow-Fast-Fast Resonant Interactions

Without loss of generality we assume that $\vec{k}'$ and $\vec{k}$ are the fast modes and $\vec{k}''$ is the slow mode. In this case the three-wave resonance condition in Eq. (3.24) reduces to

$$\begin{aligned} &\vec{k}' + \vec{k}'' = \vec{k} \\ &\omega(\vec{k}') = \omega(\vec{k}) \ , \quad \text{where} \quad \omega(\vec{k}) = \frac{(|\vec{k}_H|^2 + F^2 k_3^2)^{1/2}}{|\vec{k}|} \ . \end{aligned} \tag{4.27}$$

It is straightforward to verify that the solutions $\vec{k}'$ of the resonance equation in (4.27) lie in the cone generated by rotating $\vec{k}$ around the vertical axis. To compute the quadratic interaction coefficient in Eq. (3.12) we utilize the eigenvectors in Eqs. (4.24), (4.25) and (4.26), so that there are several cases to consider. If one of the fast modes has non-zero horizontal wave number, so does the other, that is, $\vec{k}_H \neq 0$, then $\vec{k}'_H \neq 0$. If the slow mode has non-zero horizontal wave number, $\vec{k}''_H \neq 0$, then the quadratic interaction coefficient is given by

$$\begin{aligned} B^{(\pm 1,0,\pm 1)}_{(\vec{k}',\vec{k}'',\vec{k})} = &\frac{1}{4\omega^2(\vec{k})\omega(\vec{k}'') \left|\vec{k}'_H\right| \left|\vec{k}'\right| \left|\vec{k}''\right| \left|\vec{k}_H\right| \left|\vec{k}\right|} \Bigg\{ \\ &(\vec{k}'^{\perp}_H \cdot \vec{k}''_H) \Big[F^2(|\vec{k}'_H|^2 - |\vec{k}''_H|^2)k'_3 k_3 + F^2|\vec{k}_H|^2 k'_3 k''_3 + |\vec{k}'_H|^2|\vec{k}_H|^2 \\ &+ \omega(\vec{k})^2 \Big[2(\vec{k}'_H \cdot \vec{k}''_H)k'_3 k_3 + |\vec{k}'_H|^2(k'^2_3 - k''^2_3) + |\vec{k}'_H|^2|\vec{k}_H|^2\Big]\Big] \\ &\mp i \Big[3(\vec{k}'^{\perp}_H \cdot \vec{k}''_H)^2 k'_3 k_3 + [(\vec{k}'_H \cdot \vec{k}''_H)k'_3 - \left|\vec{k}'_H\right|^2 k''_3][(\vec{k}''_H \cdot \vec{k}_H)k_3 - \left|\vec{k}_H\right|^2 k''_3]\Big]\Bigg\} \end{aligned} \tag{4.28}$$

On the other hand, if $\vec{k}''_H = 0$ then the quadratic interaction coefficient is given by

$$B^{(\pm1,0,\pm1)}_{(\vec{k}',\vec{k}'',\vec{k})} = \pm\frac{\left|\vec{k}_H\right|^2 k_3''}{4\omega(\vec{k})\left|\vec{k}'\right|\left|\vec{k}\right|} \tag{4.29}$$

and this corresponds physically to Slow-Fast-Fast interactions generated by hydrostatic balanced vertical density variations. For any of the remaining case of two fast rotating waves with vertical wave numbers $\vec{k}_H = \vec{k}'_H = 0$, then the quadratic interaction coefficient is zero. In particular, the fast vertical shear modes in (4.25) never participate in any nontrival Slow-Fast-Fast interactions.

Fast-Fast-Fast Resonant Interactions

The condition neccesary for the resonant interaction of three fast (gravity) waves are

$$\begin{gathered} \vec{k}' + \vec{k}'' = \vec{k} \\ \omega(\vec{k}') \pm \omega(\vec{k}'') = \omega(\vec{k}) \end{gathered} \tag{4.30}$$

where $\omega(\vec{k}) = (|\vec{k}_H|^2 + F^2k_3^2)^{1/2}/|\vec{k}|$. The solution of the resonance equations in (4.30) is very involved algebraically and may not have solutions for some ranges in the ratio $F = Fr/Ro$. For example, when the Rossby and the Froude number are close then $F \approx 1$ and there will not be any solutions of the resonant equations in (4.30). Moreover, although the formula for the quadratic interaction coefficient $B^{(\alpha',\alpha'',\alpha)}_{(\vec{k}',\vec{k}'',\vec{k})}$ is given explicitely in Eq. (3.12), the actual evaluation for the general formula in terms of the eigenvectors in Eq. (4.24) is cumbersome. Instead, we want to concentrate here on specific examples involving the uniformly rotating vertical shear flows.

We know from section 3 that in the low Fr and fixed Ro regime these vertical shear flows were slow modes, but in the present regime of low Fr and low Ro number they become fast rotating waves with the inertial frequency of rotation $\epsilon^{-1}F$. We are interested in demonstrating that these waves are active participants in Fast-Fast-Fast interactions. Since a vertical shear flow $\vec{V}_H(x_3,t)$ is a superposition of modes $\vec{r}^{\,(\pm1)}_{(\vec{k})}$ with $\vec{k} = (0,0,k_3)$, we consider resonant interactions involving a vertical wave number. To be more specific, we discuss next the case of a Fast-Fast-Fast interaction where a vertical wave number shear flow resonantly interacts with other gravity wave modes

We let $\vec{k}' = ({k_1}', {k_2}', {k_3}')$ and $\vec{k}'' = (-{k_1}', -{k_2}', {k_3}'')$ so that with the vertical wave number $\vec{k} = (0,0,k_3)$, the three wave resonant equations become

$$\vec{k}' + \vec{k}'' = \vec{k}$$

$$\left(\frac{|\vec{k}'_H|^2 + F^2k_3'^2}{|\vec{k}'_H|^2 + k_3'^2}\right)^{1/2} - \left(\frac{|\vec{k}'_H|^2 + F^2k_3''^2}{|\vec{k}'_H|^2 + k_3''^2}\right)^{1/2} = F \tag{4.31}$$

The second equation has a solution if

$$|k_3| < \left(\frac{F(2-F)}{1-2F}\right)^{1/2} |\vec{k}'_H|$$

and the solution k'_3, k''_3 can be integers for appropriate values of F. In this case the quadratic interaction coefficient is

$$B^{(1,-1,1)}_{(\vec{k}',\vec{k}'',\vec{k})} = \frac{1}{4\sqrt{2}} \frac{(k'_1 - ik'_2)}{|\vec{k}'||\vec{k}''|} \left(\frac{\omega(\vec{k}'')}{\omega(\vec{k}')} k'_3 - \frac{\omega(\vec{k}')}{\omega(\vec{k}'')} k''_3\right). \tag{4.32}$$

and this quadratic coefficient is zero only when the expression in parenthesis in (4.32) is zero. But this term vanishes if and only if $k'_3 = k''_3 = k_3/2$, i.e. when

$$\vec{k}' = (k'_1, k'_2, k_3/2)$$

$$\vec{k}'' = (-k'_1, -k'_2, k_3/2).$$

These examples show that the vertical shear modes can be generated through other fast wave interactions in the low Rossby number and low Froude number limit; in the low Froude number and finite Rossby number limit described in section 3, such vertical shear modes are part of the slow dynamics and can never be generated through any resonant three-wave interaction of slow or fast modes.

5. COMPARISON OF LOW FROUDE NUMBER, FINITE ROSSBY NUMBER LIMITING DYNAMICS WITH LOW FROUDE NUMBER, LOW ROSSBY NUMBER LIMITING DYNAMICS

The quasigeostrophic slow dynamics equations are given by

$$\begin{aligned} &\vec{v}_H = \nabla_H^{\perp}\psi \\ &q = \Delta_H \psi + F^2 \frac{\partial^2 \psi}{\partial x_3^2} \\ &\frac{\partial q}{\partial t} + J_H(\psi, q) = (Re)^{-1}\Delta\left(\Delta_H\psi + (Pr)^{-1}F^2\frac{\partial^2\psi}{\partial x_3^2}\right) \end{aligned} \tag{5.1}$$

where $J_H(\psi,\omega) = \nabla_H^{\perp}\psi \cdot \nabla_H\,\omega$, the Jacobian of ψ and ω in the horizontal variables.

On the other hand, in the low Fr, fixed Ro, slow limiting dynamics, the horizontal velocity $\vec{v}_H$ is given by

$$\vec{v}_H = \vec{V}_H + \nabla_H^{\perp}\psi$$
$$\omega = \Delta_H \psi \tag{5.2}$$

where the vertical shear $\vec{V}_H$, the vertical vorticity ω and the stream function ψ satisfy

$$\begin{aligned}
&\frac{\partial}{\partial t}\vec{V}_H + (Ro)^{-1}\vec{V}_H^{\perp} = (Re)^{-1}\frac{\partial^2}{\partial x_3^2}\vec{V}_H \\
&\frac{\partial \omega}{\partial t} + \vec{V}_H \cdot \nabla_H \, \omega + J_H(\psi, \omega) = (Re)^{-1}\Delta_H \omega + (Re)^{-1}\frac{\partial^2 \omega}{\partial x_3^2} \\
&\Delta_H \psi = \omega \ .
\end{aligned} \tag{5.3}$$

The quasigeostrophic slow dynamics equations in (5.1) and the low Fr, fixed Ro limiting dynamics in (5.2) and (5.3) are strikingly different. The most crucial difference is the presence of the vertical shear flow $\vec{V}_H(x_3, t)$ in Eq. (5.2). This flow is completely absent in the quasigeostrophic equations in (5.1). Indeed, we know that the vertical shear $\vec{V}_H$ is generated by the vertical wave number modes associated with $\vec{r}^{\,(\pm 1)}_{(\vec{k})}$ in Eq. (3.5). However, in the low Fr, low Ro approximation these modes no longer represent slow vortical modes, but instead contribute to the fast limiting dynamics as fast rotation modes with the inertial frequency, $\epsilon^{-1}F$.

This fact has important implications regarding the limiting slow dynamics in both regimes. In the limiting dynamic equations for low Fr and fixed Ro given in Eq. (5.3), the vertical vorticity ω is advected by the vertical shear $\vec{V}_H$. Therefore the vertical shear will intensify the gradients of ω in the vertical direction, and this will induce substantial additional dissipation in the vertical direction.

To emphasize this point, we recall that the parameter F in Eq. (5.1) is the ratio of the Froude and Rossby numbers, $F = Fr/Ro$. Therefore, the small F limit corresponds to the regime where the buoyancy time is much smaller than the rotation time. Utilizing the standard techniques developed in Klainerman and Majda (1981), and Majda (1984), it is possible to show rigorously that the small F limit of the quasigeostrophic equations in (5.1) exists, but it is not the complete system given by Eqs. (5.2) and (5.3). In fact, the low F limit is the particular case of the slow dynamics equations (5.2) and (5.3) where the vertical shear modes $\vec{V}_H(x_3, t)$ are not excited, i.e., $\vec{V}_H \equiv 0$. Furthermore, it is evident from (5.3) that the vertical shear modes can never be actively created by any mechanism in the low Froude number and finite Rossby number limit. In contrast, in the low Rossby number and low Froude number limit, it is established in (4.31) and (4.32) that the vertical shear modes which are fast waves in this context can be created easily through three resonant three-wave interactions of fast gravity waves; on the other hand, it is readily verified that these fast vertical shear modes can never be created from Slow-Fast-Fast interactions in this limit.

Effect of Varying Prandtl Number

Another difference in the slow dynamics is the anisotropic dependence of the diffusion on the Prandtl number in (5.1). For $Pr \neq 1$ there is anisotropic diffusion in the quasigeostrophic slow dynamics equation for the potential vorticity q. This reflects the fact that in the quasigeostrophic approximation the density stratification plays a role in the diffusion of potential vorticity, where the variations in the density ρ are related to the diffusion of ψ through Eq. (4.10). On the other hand, the limiting slow dynamics for low Fr and fixed Ro have no dependence on Pr. This is due to the fact that in this asymptotic regime the evolution of the density ρ and the velocity $\vec{v}_H$ are decoupled, and therefore it is to be expected that changes in the diffusive properties of the density would have no effect on the evolution of the velocity field.

To quantitatively assess the finite Prandtl number effects in the quasigeostrophic equations, we linearize the equations in (5.1) and calculate that the solutions are damped by diffusion at the wave number $\vec{k}$ at the rate

$$-(Re)^{-1}|\vec{k}|^2 \left(\frac{|\vec{k}_H|^2 + F^2(Pr)^{-1}k_3^2}{|\vec{k}_H|^2 + F^2k_3^2} \right).$$

The expression in parenthesis is less than one for $Pr > 1$ so less laminar dissipation occurs in the equations in (5.1) compared with (5.2) under these circumstances. The situation is reversed for $Pr < 1$.

Next we consider the Prandtl number effects on diffusion for the reduced fast dynamics. The diffusion coefficients $D^{(\pm1,\pm1)}_{(\vec{k})}$ are given in Eq. (3.22) and Eq. (3.23). For the low Fr, low Ro limiting dynamics equations, the explicit values of the diffusion coefficients associated with the fast waves are

$$D^{(\pm1,\pm1)}_{(\vec{k})} = -(Re)^{-1}\frac{|\vec{k}|^2}{2} \left\{ \frac{(1+(Pr)^{-1})|\vec{k}_H|^2 + 2F^2k_3^2}{|\vec{k}_H|^2 + F^2k_3^2} \right\} \tag{5.4}$$

while for the low Fr, fixed Ro limiting dynamics

$$D^{(\pm1,\pm1)}_{(\vec{k})} = -(Re)^{-1}\frac{|\vec{k}|^2}{2} \left\{ 1 + (Pr)^{-1} \right\}. \tag{5.5}$$

These equations show the dependence of the diffusion on the Prandtl number in both the asymptotic regime of low Fr and low Ro numbers, and the asymptotic regime of low Fr and fixed Ro numbers. Notice that for large values of the Pr number the magnitude of the coefficients is reduced; for example, in the low Fr, fixed Ro regime and in the limit of $Pr \to \infty$, the laminar diffusive damping on the fast waves is half what is experienced by the slow modes. The relative magnitude of the diffusion coefficients in Eqs. (5.4) and (5.5) also changes depending on whether $Pr < 1$ or $Pr > 1$. This can be seen by comparing the expressions in braces for both Eqs. (5.4) and (5.5). For $Pr < 1$

the laminar diffusive damping in the low Fr, fixed Ro numbers regime is larger than in the low Fr, low Ro regime; the opposite holds when $Pr > 1$. When $Pr = 1$ both coefficients are equal to the Reynolds diffusion.

Elementary solutions of the reduced dynamics equations

Example 1. Planar solutions of the slow dynamics.
By a planar solution we mean a solution of the 3-D limiting dynamics equations that depends only on the vertical and one of the horizontal space variables. More precisely, let $\vec{n}$ be a horizontal unit direction and define the plane wave variable n by

$$n = \vec{n} \cdot \vec{x}_H \tag{5.6}$$

then a planar solution is a function of n, x_3 and t. Now it is straightforward to check that if $\psi = \psi(\vec{n} \cdot \vec{x}_H, x_3, t)$, and $q = q(\vec{n} \cdot \vec{x}_H, x_3, t)$, then the horizontal Jacobian $J_H(\psi, q) = 0$, and the potential vorticity equation in (5.1) reduces to a linear equation. Moreover, if $Pr = 1$, then the potential vorticity equation in (5.1) reduces to the standard heat equation

$$\frac{\partial q}{\partial t} = (Re)^{-1} \left(\frac{\partial^2 q}{\partial n^2} + \frac{\partial^2 q}{\partial x_3^2} \right) \tag{5.7}$$

For the same reason, if we assume a planar vertical vorticity $\omega = \omega(\vec{n} \cdot \vec{x}_H, x_3, t)$ for the low Fr, fixed Ro limiting dynamics in Eq. (5.2) and Eq. (5.3), then $J_H(\psi, \omega) = 0$, and the limiting dynamics in Eq. (5.3) reduces to a linear system of equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\vec{V}_H + (Ro)^{-1}\vec{V}_H^{\perp} = (Re)^{-1}\frac{\partial^2}{\partial x_3^2}\vec{V}_H \\
&\frac{\partial \omega}{\partial t} + \vec{V}_H \cdot \vec{n}\frac{\partial \omega}{\partial n} = (Re)^{-1}\left(\frac{\partial^2 \omega}{\partial n^2} + \frac{\partial^2 \omega}{\partial x_3^2}\right) \\
&\psi = \frac{\partial^2}{\partial n^2}\omega,
\end{aligned} \tag{5.8}$$

In the quasigeostrophic equation in (5.7) the dissipation of potential vorticity is due purely to diffusive effects. On the other hand, in the slow dynamics equations in (5.8) the vertical shear modes can induce substantial additional dissipation of the vertical vorticity through enhanced dissipation in the horizontal direction created by the shearing motion through the purely linear advection-diffusion in (5.8). These are the simplest solutions exhibiting the effects of shear in a transparent fashion for (5.8) with such effects completely absent for the planar quasigeostropic flows in (5.7).

Example 2. Simple examples of Slow-Fast-Fast interaction for the reduced dynamics.
A simple way to incorporate Slow-Fast-Fast interactions into the reduced dynamics for both the regime of low Fr, low Ro numbers , and the regime of low Fr and fixed Ro number, consists in taking initial data lying on the same cone. This can be done as follows. Consider a wave number vector $\vec{k}^0 = (\vec{k}^0_H \ , k^0_3)$, with $\vec{k}^0_H = (k^0_1 \ , \ k^0_2)$. Now define another vector $\vec{k}^{0\,\perp}$ in the same cone containing $\vec{k}^0$, $\vec{k}^{0\,\perp} = (\vec{k}^{0\,\perp}_H \ , k^0_3)$, where $\vec{k}^{0\perp}_H = (-k^0_2 \ , \ k^0_1)$ is orthogonal to $\vec{k}^0_H$. With these two vectors $\vec{k}^0$ and $\vec{k}^{0\perp}_H$ generate the cones

$$
\begin{aligned}
C_0 &= \left\{ j(\pm\vec{k}^0_H \ , \ k^0_3) | \ j = \pm 1, \pm 2, \ldots \right\} \\
C_0^{\perp} &= \left\{ j(\pm\vec{k}^{0\,\perp}_H \ , \ k^0_3) | \ j = \pm 1, \pm 2, \ldots \right\} .
\end{aligned}
\tag{5.9}
$$

Then it is clear that the modes in the set $C_0 \cup C_0^{\perp}$ generate the Fast-Fast modes in the Slow-Fast-Fast resonant interactions for either the low Fr, low Ro regime, or the low Fr and finite Ro regime; the reason is that the above set is contained in the cone generated by rotating $\vec{k}^0$ around the vertical axis. In addition, this set alone does not generate Fast-Fast-Fast resonant interactions. The slow vortical modes that resonate with the fast modes from the set $C_0 \cup C_0^{\perp}$ are given by the set S_0

$$
S_0 = \left\{ (p(\pm\vec{k}^0_H) - q(\pm\vec{k}^{0\,\perp}_H) \ , \ (p-q)k^0_3) | \ p, q = \pm 1, \pm 2, \ldots \right\} . \tag{5.10}
$$

For example, suppose now that the solution of the slow dynamics equations is given by one of the planar waves from Example 1, with $\vec{n} = (1, 0)$ for simplicity. Then, in either the low Fr, fixed Ro, or the low Fr, low Ro regime, the set of non-zero slow vortical modes for the planar solution involves the wave numbers

$$
S_P = \{ (k_1, 0, k_3) | k_1, k_3 = 0, \pm 1, \pm 2, \ldots \} . \tag{5.11}
$$

In order to have Slow-Fast-Fast interactions involving the planar solution in the direction of $\vec{n} = (1, 0)$ and the fast modes from the set $C_0 \cup C_0^{\perp}$, it is neccesary that the set S_0 in Eq. (5.10) and the set S_P in Eq. (5.11) have non empty intersection. This is the case if the equation

$$
pk^0_2 \pm qk^0_1 = 0 \tag{5.12}
$$

with solutions $p = jk^0_1$, and $q = \mp jk^0_2$ is satisfied. ¿From Eqs. (5.4) and (5.5) the rates of dissipation of the fast modes is proportional to $|\vec{k}|^2$. Since the fast modes are $p(\pm\vec{k}^0_H, k^0_3)$ and $q(\pm\vec{k}^{0\,\perp}_H, k^0_3)$, then the rate of dissipation is proportional to p^2 and q^2. If these numbers are very disparate, for example, $|p| \gg |q|$, then one of the resonant fast modes is strongly dissipative and the other fast mode is weakly dissipative. Since the solutions p and q of Eq. (5.12) are $p = jk_1^0$, and $q = \mp jk_2^0$, the condition $|p| \gg |q|$ is equivalent to have $|k_1^0| \gg |k_2^0|$.

6. CONCLUDING DISCUSSION

We have presented a mathematically rigorous framework for fast wave averaging for idealized geophysical flows in periodic geometry and have applied this theory to derive reduced low Froude number limiting dynamics in strongly stratified rotating flows with either fixed or small Rossby numbers. The theory described in section 2 applies in general for only order one advective time scales with the precise interval of validity depending on the temporal growth of the first three derivatives of the solutions (Majda, 1984); in particular, additional secular terms in time can arise in solutions for longer time scales of order $O(\epsilon^{-1})$. Obviously this is an important and interesting research direction.

What is the ultimate source for the differences in limiting slow dynamics at low Froude numbers as compared with those for low Froude and low Rossby numbers? This can be traced back to the differences in the dispersion relations for the fast waves in the two situations described earlier in (2.22)-(2.25) since both slow dynamics include vortical perturbations. In the low Froude number, finite Rossby number limit, the fast (gravity) wave dispersion relation, $\Omega = \pm|\vec{k}_H|/|\vec{k}|$, is degenerate and vanishes at wave numbers with $|\vec{k}_H| = 0$ giving rise to additional vertically sheared horizontal motions as part of the slow dynamics. On the other hand, in the low Rossby number and low Froude number limit, the fast (gravity) wave dispersion relation, Ω, from (2.25) never vanishes for any wave number and is uniformly bounded away from zero so all gravity waves are in the leading order fast dynamics and the space of states for the slow dynamics is smaller.

The systematic averaging procedure also leads to general reduced dynamics with solutions involving Slow-Fast-Fast and Fast-Fast-Fast three-wave resonant interactions riding along with the evolving slow dynamics. For applications involving turbulent stratified flows, one might envision a statistical theory for energy transfer based on such resonant interactions. There already has been some recent work along these lines. The study by Bartello (1995) for Slow-Fast-Fast interactions is based upon equilibrium statistical spectra while Waleffe (1992, 1993) has developed a statistical closure theory based on transfer mechanisms for resonant triads for uniformly rotating flows; Godeferd and Cambon (1994) have utilized an anisotropic (E.D.Q.N.M.) closure theory in a similar spirit for stratified flows.

Another feature which we have emphasized in our systematic study here is the effect of varying Prandtl numbers which is discussed in detail in section 5. We do this in this paper since laboratory experiments with strongly stratified flows have Prandtl numbers on the order of 200 while the idealized numerical simulations assume $Pr = 1$ and furthermore the oceans also have large Prandtl numbers. As regards actual geophysical flows, this allows us to understand anisotropic vector eddy diffusivity effects by varying the Prandtl number and our arguments readily extend to other higher order forms of dissipation.

We briefly discuss the effects of varying Prandtl numbers described in section 5 in a more general context. For the limiting dynamics with low Froude numbers and finite Rossby numbers, the fast gravity waves decay at nearly one-half the rate of the slow vortical modes at a given wave number when $Pr \gg 1$. This slower decay rate might seriously alter the energy ratio of gravity modes to vortical modes at low Froude numbers in decaying simulations in periodic geometry compared with $Pr = 1$ (Metais and Herring, 1989; Bartello, 1995) or perhaps this effect might be swamped by more rapid energy transfer of gravity waves through catalytic Slow-Fast-Fast interactions (Bartello, 1995). This is clearly an interesting topic for further investigation. Of course in laboratory experiments where the walls are distant, such reduced gravity wave decay rates at large Prandtl numbers can be nearly negligible due to radiation.

For the low Rossby number and low Froude number limiting dynamics, and $Pr \gg 1$, the situation is more subtle. From (5.4) and the preceding formula in that section, the slow quasigeostrophic modes decay faster than the fast gravity modes for wave numbers $\vec{k}$ with

$$|\vec{k}_H|^2 > 2F^2 k_3^2$$

and slower than the gravity modes in the opposite cone of wave numbers

$$|\vec{k}_H|^2 < 2F^2 k_3^2$$

i.e. for larger scale horizontal motions compared with the vertical.

Finally, we mention that throughout this work, we have emphasized the role of three wave resonant interactions including the Fast-Fast-Fast (internal gravity) three wave resonances (McComas and Bretherton, 1977; Muller et al., 1986). Recently, Babin et al. (1996) have developed another approach to fast wave averaging where such three-wave resonant interactions are eliminated by hypothesis at the outset and sophisticated mathematical theories of small divisors, first introduced by Schochet (1994) in a similar context, are utilized.

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APPENDIX

Here we demonstrate that in the low Fr, low Ro regime of the stably stratified Boussinesq equations with slanted rotation, the potential vorticity has no fast wave contributions to leading order in $\epsilon = Fr$. The leading term q^0 of the potential vorticity in Eq. (2.21) is given by

$$q^0 = \omega_3 - F(\eta \cdot \nabla \rho) \ . \tag{A.1}$$

where ω_3 is the vertical component of the vorticity. Let $\vec{u} = e^{i(\vec{k}\cdot\vec{x} \mp \omega(\vec{k})t)} \vec{r}\,^{(\pm 1)}_{(\vec{k})}$ be a fast gravity mode, where $\omega(\vec{k}) = \left(F^2(\vec{\eta}\cdot\vec{k})^2 + |\vec{k}_H|^2\right)^{1/2} / |\vec{k}|$, and $\vec{r}\,^{(\pm 1)}_{(\vec{k})} = ^t (\vec{v}^{(\pm 1)} \ , \ \rho^{(\pm 1)})$, then its contribution to q^0 is

$$q^0 = i|\vec{k}| \left[\vec{v}_H^{(\pm 1)} \cdot \tilde{k}_H^{\perp} - F(\eta \cdot \tilde{k})\rho^{(\pm 1)}\right] e^{i(\vec{k}\cdot\vec{x} \mp \omega(\vec{k})t)} \ , \tag{A.2}$$

where $\tilde{k} = (\tilde{k}_H \ , \tilde{k}_3) = \vec{k}/|\vec{k}|$ is the normalized wave number, and $\tilde{k}_H^{\perp} = (-\tilde{k}_2, \tilde{k}_1)$. To prove that q^0 vanishes now reduces to verify the following algebraic identity

$$Q^0 = \vec{v}_H^{(\pm 1)} \cdot \tilde{k}_H^{\perp} - F(\eta \cdot \tilde{k})\rho^{(\pm 1)} = 0 \tag{A.3}$$

for the right eigenvector $\vec{r}\,^{(\pm 1)}_{(\vec{k})}$ associated with the fast mode. A lengthy but straightforward algebraic calculation yields these eigenvectors for the slanted case

$$\vec{r}\,^{(\pm 1)}_{(\vec{k})} = \begin{pmatrix} \mathcal{D}_\pm^{-1}\left\{F\left[\omega^2(\eta_2 + \tilde{k}_1\tilde{\alpha}_3) + \eta_3\tilde{k}_2\tilde{k}_3\right] - (\pm i\omega)(F^2(\vec{\eta}\cdot\tilde{k})\eta_1\tilde{k}_3 + \tilde{k}_1\tilde{k}_3)\right\} \\ \mathcal{D}_\pm^{-1}\left\{F\left[\omega^2(-\eta_1 + \tilde{k}_2\tilde{\alpha}_3) - \eta_3\tilde{k}_1\tilde{k}_3\right] - (\pm i\omega)(F^2(\vec{\eta}\cdot\tilde{k})\eta_2\tilde{k}_3 + \tilde{k}_2\tilde{k}_3)\right\} \\ -(\pm i\omega) \\ 1 \end{pmatrix} \tag{A.4}$$

where $\omega = \omega(\vec{k})$, $\tilde{\alpha} = \vec{\eta} \times \tilde{k}$, and $\mathcal{D}_\pm$ is given by

$$\mathcal{D}_\pm = -\omega^2 - (\pm i\omega)F(\tilde{k}_H \cdot \tilde{\alpha}_H) + F^2(\eta \cdot \tilde{k})\eta_3\tilde{k}_3 \tag{A.5}$$

Introducing $\vec{r}\,^{(\pm 1)}_{(\vec{k})}$ back into Eq. (A.3) yields after some manipulation

$$\begin{aligned} Q^0 = \mathcal{D}_\pm^{-1}\left\{F\left[-\omega^2(\eta_H \cdot \tilde{k}_H) - |\tilde{k}_H|^2\eta_3\tilde{k}_3\right] - (\pm i\omega)F^2(\vec{\eta}\cdot\tilde{k})(\eta_H \cdot \tilde{k}_H^{\perp})\tilde{k}_3\right\} \\ - F(\eta \cdot \tilde{k}) \ . \end{aligned} \tag{A.6}$$

To further simplify Eq. (A.6) we utilize the algebraic identities

$$-\omega^2(\eta_H \cdot \tilde{k}_H) - |\tilde{k}_H|^2\eta_3\tilde{k}_3 = -\omega^2(\eta \cdot \tilde{k}) + F^2(\eta \cdot \tilde{k})^2\eta_3\tilde{k}_3$$
$$\eta_H \cdot \tilde{k}_H^{\perp} = -\tilde{\alpha}_3 \tag{A.7}$$

so that Eq. (A.6) reduces to

$$Q^0 = \mathcal{D}_{\pm}^{-1}\left\{-\omega^2 + F^2(\eta \cdot \tilde{k})\eta_3\tilde{k}_3 + (\pm i\omega)F\tilde{\alpha}_3\tilde{k}_3\right\}F(\eta \cdot \tilde{k}) - F(\eta \cdot \tilde{k}) \; . \tag{A.8}$$

Utilizing the formula for $\mathcal{D}_{\pm}$ in Eq. (A.5), we finally conclude that

$$Q^0 = \vec{v}_H^{(\pm 1)} \cdot \tilde{k}_H^{\perp} - F(\eta \cdot \tilde{k})\rho^{(\pm 1)} = \mathcal{D}_{\pm}^{-1}\mathcal{D}_{\pm}F(\eta \cdot \tilde{k}) - F(\eta \cdot \tilde{k}) = 0 \; . \tag{A.9}$$

This proves that the leading term q^0 of the potential vorticity does not contain fast gravity waves.

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Table 1. Examples of Fast-Fast-Fast resonant interactions for the low Froude and finite Rossby numbers regime. The wave numbers $\vec{k}'$, $\vec{k}''$, $\vec{k}$ satisfy Eq. (3.44) and the quadratic interaction coefficient $B^{(1,1,1)}_{(\vec{k}',\vec{k}'',\vec{k})}$ is given by Eq. (3.45).

$\vec{k}'$	$\vec{k}''$	$\vec{k} = \vec{k}' + \vec{k}''$	$B^{(1,1,1)}_{(\vec{k}',\vec{k}'',\vec{k})}$
$(\pm5, 0, 10)$	$(\pm3, 0, -6)$	$(\pm8, 0, 4)$	1.897367
$(0, \pm5, 10)$	$(0, \pm3, -6)$	$(0, \pm8, 4)$	1.897367
$(\pm5, 0, -10)$	$(\pm3, 0, 6)$	$(\pm8, 0, -4)$	−1.897367
$(0, \pm5, -10)$	$(0, \pm3, 6)$	$(0, \pm8, -4)$	−1.897367
$(\pm16, \pm8, -12)$	$(\mp2, \mp1, 16)$	$(\pm14, \pm7, 4)$	6.622539
$(\pm16, \mp8, -12)$	$(\mp2, \pm1, 16)$	$(\pm14, \mp7, 4)$	6.622539
$(\pm8, \pm16, -12)$	$(\mp1, \mp2, 16)$	$(\pm7, \pm14, 4)$	6.622539
$(\mp8, \pm16, -12)$	$(\pm1, \mp2, 16)$	$(\mp7, \pm14, 4)$	6.622539
$(\pm16, \pm8, 12)$	$(\mp2, \mp1, -16)$	$(\pm14, \pm7, -4)$	−6.622539
$(\pm16, \mp8, 12)$	$(\mp2, \pm1, -16)$	$(\pm14, \mp7, -4)$	−6.622539
$(\pm8, \pm16, 12)$	$(\mp1, \mp2, -16)$	$(\pm7, \pm14, -4)$	−6.622539
$(\mp8, \pm16, 12)$	$(\pm1, \mp2, -16)$	$(\mp7, \pm14, -4)$	−6.622539
$(\pm2, \pm2, 19)$	$(\mp10, \mp10, -13)$	$(\mp8, \mp8, 6)$	4.268749
$(\pm2, \mp2, 19)$	$(\mp10, \pm10, -13)$	$(\mp8, \pm8, 6)$	4.268749
$(\pm2, \pm2, -19)$	$(\mp10, \mp10, 13)$	$(\mp8, \mp8, -6)$	−4.268749
$(\pm2, \mp2, -19)$	$(\mp10, \pm10, 13)$	$(\mp8, \pm8, -6)$	−4.268749
$(\pm10, 0, 20)$	$(\pm6, 0, -12)$	$(\pm16, 0, 8)$	3.794733
$(0, \pm10, 20)$	$(0, \pm6, -12)$	$(0, \pm16, 8)$	3.794733
$(\pm10, 0, -20)$	$(\pm6, 0, 12)$	$(\pm16, 0, -8)$	−3.794733
$(0, \pm10, -20)$	$(0, \pm6, 12)$	$(0, \pm16, -8)$	−3.794733