PDE, Spring 2020, HW6. Distributed Friday 5/1/2020, due by 9pm Friday 5/15/2020 (no extensions – I will distribute a solution sheet on that evening). Upload your solution using the Assignments tool in NYU Classes; if possible, please provide a single pdf.

Correction 5/15: in Problem 3(c), in the two-part characterization of a viscosity solution, I forgot the condition that $\phi(x_0) = u(x_0)$; equivalently, I could have written for the first bullet that if $u - \phi$ has a local maximum at x_0 then $\lambda u + H(\nabla\phi) \leq f$ at x_0, and for the second bullet that if $u - \phi$ has a local maximum at x_0 then $\lambda u + H(\nabla\phi) \geq f$ at x_0.

Correction 5/13: for Problem 1 one needs some condition on f to be sure $f(u) \in L^2$. I have fixed this by adding the condition $f(0) = 0$.

(1) In Lecture 12, we proved a local-in-time existence result for

$$u_{tt} - \Delta u = f(u) \quad \text{in } \mathbb{R}^n\text{, with } u = g \text{ and } u_t = h \text{ at } t = 0,$$

provided $g \in H^k$ and $h \in H^{k-1}$ with $k > n/2$, obtaining a solution in $u \in L^\infty(0, T; H^k)$ with $u_t \in L^\infty(0, T; H^{k-1})$ for some $T > 0$. (Throughout this problem, I write H^k for $H^k(\mathbb{R}^n)$. The proof relied on the fact that if f is C^k and $f(0) = 0$, and $u \in H^k$ with $k > n/2$, then $f(u) \in H^k$ and there is an estimate of the form

$$\|f(u)\|_{H^k} \leq \phi\left(\|u\|_{H^k}\right)$$

where ϕ is a suitable continuous, nondecreasing function of its argument. Prove this result. (Note: In Section 12.2.2 of Evans you'll find a more general result, with part of the proof given explicitly and the rest left as an exercise for which he gives a hint. The arguments and guidance offered there are of course relevant here as well, though the case stated here is perhaps a little simpler to write down.)

(2) We noted in Lecture 12 that for an equation of the form

$$u_{tt} - \Delta u = -|u|^{p-1}u \quad \text{in } \mathbb{R}^n$$

with $p > 1$, a smooth enough solution (with sufficient decay at infinity to justify the integration by parts $\int_{\mathbb{R}^n} \nabla u \cdot \nabla u_t \, dx = -\int_{\mathbb{R}^n} u_t \Delta u \, dx$) has

$$\frac{d}{dt} \int_{\mathbb{R}^n} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{p+1}|u|^{p+1} \, dx = 0,$$

which gives uniform-in-time bounds for $\int u_t^2 \, dx$, $\int |\nabla u|^2 \, dx$, and $\int |u|^{p+1} \, dx$. Do these bounds prevent u from blowing up in finite time? This question is more subtle than it looks: Evans gives affirmative answers for $n = 3, p < 5$ and $n = 3, p = 5$ in Sections 12.3.3 and 12.4 respectively, but the arguments are fairly subtle. The case $n = 3, p \leq 3$ is easier, as this problem shows.[1]

[1] To avoid technicalities, you may assume in doing this problem that u has compact support in space at every time, and it is regular enough to permit the desired calculations. In practice, if the initial data for u and u_t have compact support then $u(t)$ has compact support for all t since information propagates at finite speed (we proved this in Lecture 12). If in addition $u \in H^2$ and $u_t \in H^1$ initially, then since $2 > 3/2$ and $f(u) = |u|^{p-1}u$ is continuously differentiable at 0 for $p > 1$, the local-in-time existence theory we did in Lecture 12 shows, in space dimension 3, existence and uniqueness of a solution $u(t) \in H^2$ with $u_t \in H^1$ on a maximal interval $[0, T^*]$, where either one of these norms blows up as $t \to T^*$ or else $T^* = \infty$. Alternatively, the arguments for (b)–(d) can be replicated with v being a difference quotient rather than a derivative of u; this gives the desired results in the limit $\Delta x \to 0$.

(a) Show that the L^2 norm of u grows at most linearly in time.

(b) Let $v = \partial u/\partial x_j$ for some j, and observe that $v_{tt} - \Delta v = -p|u|^{p-1}v$. Show that $E(t) = \int_{\mathbb{R}^3} \frac{1}{2}v_t^2 + \frac{1}{2}|\nabla v|^2\, dx$ satisfies

$$\frac{dE}{dt} \leq p\left(\int v_t^2\, dx\right)^{1/2}\left(\int |u|^{2(p-1)}v^2\, dx\right)^{1/2}$$

(c) To handle the case $p = 3$, show that

$$\int |u|^4 v^2\, dx \leq C\left(\int |\nabla u|^2\right)^2\left(\int |\nabla v|^2\right),$$

and conclude an estimate of the form $E(t) \leq c_1 e^{c_2 t}$, where c_1 and c_2 depend only on the initial data for u and u_t. Thus, u remains in H^2 for all t. (Since $2 > 3/2$, this controls the L^∞ norm of u.)

(d) Adjusting appropriately what you did for part (c), show that for $1 < p < 3$, it is still true that u remains in H^2 for all t.

(3) Let $u : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$u(x) = \min_{y(0)=x} \int_0^\infty e^{-\lambda t}\left[h(dy/dt) + f(y(t))\right] dt, \tag{1}$$

where f and h are given functions and $\lambda > 0$.

(a) Show (formally) that the associated Hamilton-Jacobi-Bellman equation is

$$\lambda u + H(\nabla u) = f \tag{2}$$

with[2] $H(p) = \max_a\{a \cdot p - h(-a)\}$.

(b) If h and f are bounded and continuous, it's clear that u is well-defined. Show directly from the definition (1) that if f is Lipschitz continuous with constant M (i.e. $|f(x) - f(y)| \leq M|x - y|$) then u is Lipschitz continuous with constant M/λ.

(c) Show that u is a viscosity solution of (2), in the sense that

- if $u-\phi$ has a local maximum at x_0 and $\phi(x_0) = u(x_0)$, then $\lambda\phi + H(\nabla\phi) \leq f$ at x_0, and
- if $u-\phi$ has a local minimum at x_0 and $\phi(x_0) = u(x_0)$, then $\lambda\phi + H(\nabla\phi) \geq f$ at x_0.

(Note: this is the stationary analogue of what we did at the end of Lecture 13.)

[2] To make contact with Lecture 13, notice that eikonal equation $|\nabla u| = 1$ in Ω with $u = 0$ at $\partial\Omega$ is obtained by (i) taking $h(a)$ equal to 0 for $|a| \leq 1$ and $h = \infty$ otherwise, (ii) taking $f = 1$ in Ω, $f = 0$ outside, and (iii) setting $\lambda = 0$.

(4) When solving HJ equations in bounded domains, not only the PDE but also the boundary condition must be interpreted in an appropriate "viscosity" sense. This problem explores why there is some subtlety to the boundary condition, by considering (for a bounded domain $\Omega \subset R^2$),

$$u(x) = \min_{y(0)=x,\, |dy/dt|\leq 1} \left\{ \int_0^\tau f(y(t))\, dt + g(y(\tau)) \right\} \tag{3}$$

where τ is first time the path arrives at $\partial\Omega$.

(a) Show that the HJB equation is (formally) $|\nabla u| = f$ in Ω with $u = g$ at $\partial\Omega$.

(b) Clearly existence must fail when $f = 1$ but $|dg/ds| > 1$ somewhere on $\partial\Omega$, where dg/ds is the derivative of g with respect to arc length. Consider the example $\Omega = [-1, 1]^2$, $f = 1$, and $g(x, y) = 2(1 - |y|)$. Determine (explicitly) the optimal value $u(x)$, by solving (directly) the minimization (3).