PDE, Spring 2020, HW5. Distributed Thursday 4/9/2020, due Friday 4/24/2020 (two full weeks from distribution). Upload your solution using the Assignments tool in NYU Classes; if possible, please provide a single pdf. *Corrections and additions 4/19: I added a hint for problem 1; in Problem 2, I added the hypotheses that f is C^1 and increasing; in Problem 4(b), in the sentence starting "Your task is to show ...", I corrected a typo in the integral form of the PDE $u_t - \Delta u = u^3$; in Problem 5, I corrected the characterization of the Galerkin approximation by inserting the variable coefficient $a(x)$ where it belongs; and in Problem 6, I changed the PDE at the beginning of the problem to $u_t - \Delta u = u^3$, since that is what I had mind when writing the rest of the problem.*

(1) Let u solve $u_t - \Delta u = f(u)$ in a bounded domain Ω, with $u = 0$ at $\partial\Omega$ (and with enough smoothness to apply the maximum principle). Suppose $f(0) = 0$, and let $m \leq 0$ and $M \geq 0$ have the property that the interval $[m, M]$ is invariant for the ODE $da/dt = f(a)$ (in the sense that if $a(0) \in [m, M]$ then $a(t) \in [m, M]$ for all $t > 0$). Show that this interval is invariant for the PDE as well (in the sense that if $u(x, 0) \in [m, M]$ for all x then $u(x, t) \in [m, M]$ for all x and all $t > 0$). [Hint: to show that $u \geq m$, start by showing that $\phi = u - m$ satisfies a relation of the form $\phi_t - \Delta\phi + c(x, t)\phi \geq 0$, for a suitable function $c(x, t)$.]

(2) In Lecture 10 we made repeated use of the following Lemma: *Suppose a nonnegative real-valued function $a(t)$ satisfies a differential inequality $da/dt \leq f(a(t))$ with initial condition $a(0) = a_0$, and some C^1 function f that's strictly positive and increasing on $[a_0, \infty)$. Then $a(t) \leq \alpha(t)$ for all $t > 0$, where α solves the ODE $d\alpha/dt = f(\alpha(t))$ with the same initial data $\alpha(t) = a_0$.* Prove it.

(3) This problem guides you through a semigroup-based proof that when $f : \mathbb{R} \to \mathbb{R}$ is C^1 with $f(0) = 0$, the 1D nonlinear heat equation

$$u_t - u_{xx} = f(u) \quad \text{for } t > 0, \text{ with } u = u_0(x) \text{ at } t = 0$$

has a unique local-in-time solution in $C([0, T], H^1)$ for any $u_0 \in H^1$. (Here and throughout this problem, I write H^1 for the space $H^1(\mathbb{R})$.)

(a) Show that when $\Delta u = u_{xx}$ is the 1D Laplacian, $e^{t\Delta}$ is a bounded linear map from H^1 to itself, with operator norm at most 1 (in other words, $\|e^{t\Delta}u\|_{H^1} \leq \|u\|_{H^1}$).

(b) Show that if $u \in C([0, T], H^1)$ then

$$\int_0^t e^{(t-s)\Delta} f(u(s))\, ds \in C([0, T], H^1).$$

(c) Now consider the iteration

$$u^{n+1}(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u^n(s))\, ds,$$

with $u^0(x) = 0$. Show that if $T > 0$ is small enough the iteration converges in $C([0, T], H^1)$.

(d) Conclude that our initial value problem has a unique solution in $C([0,T],H^1)$.

(Note: the strategy outlined here amounts to an application of the contraction mapping fixed point theorem. The overall outline of the argument should be familiar from your study of ODE.)

(4) I argued in Lecture 10 (using the scale-invariance of the equation) that for the initial-value problem $u_t - \Delta u = u^3$ in $\mathbb{R}^n$, a well-posedness result in $L^p(\mathbb{R}^n)$ should need $p > n$. This problem shows that there is indeed such a well-posedness result when $p > n$.

(a) Let Δ be the Laplacian in $\mathbb{R}^n$. Show that if $u \in L^p(\mathbb{R}^n)$ then $e^{t\Delta}u \in L^q(\mathbb{R}^n)$ for $t > 0$, and

$$\|e^{t\Delta}u\|_{L^q} \le C\frac{1}{t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}}\|u\|_{L^p}$$

where C is independent of t and u. (Note: this amounts to an estimate of the operator norm $\|e^{t\Delta}\|_{L^p\to L^q}$.) [Hint: use the inequality from Real Variables: $\|f*g\|_{L^m} \le \|f\|_{L^k}\|g\|_{L^\ell}$ when $\frac{1}{k}+\frac{1}{\ell}=\frac{1}{m}+1$.]

(b) Show that the strategy of Problem 3 applies also here, for initial data $u_0 \in L^p(\mathbb{R}^n)$ with $p > n$. (Your task is to show that for sufficiently small $T > 0$, there is a unique $u \in C([0,T],L^p)$ such that $u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}u^3(s)\,ds$ for $0 \le t \le T$.)

(5) Let us examine the accuracy of a specific Galerkin scheme for the initial-value problem

$$u_t - \nabla\cdot(a(x)\nabla u) = 0 \text{ in } \Omega, \text{ with } u = 0 \text{ at } \partial\Omega \text{ and } u = u_0 \text{ at } t = 0.$$

We assume Ω is a bounded domain in $\mathbb{R}^n$ (with nice enough boundary), and take as the Galerkin space V_N the span of the first N eigenfunctions of the constant-coefficient Dirichlet Laplacian. (More carefully: let $\{\phi_j\}$ be an orthonormal basis for L^2 satisfying $-\Delta\phi_j = \lambda_j\phi_j$ in Ω and $\phi_j = 0$ at $\partial\Omega$, ordered so that $\lambda_j \le \lambda_{j+1}$; then V_N is the span of $\{\phi_j\}_{j=1}^N$.) As a reminder: the Galerkin approximation u_N is characterized by the properties that $u_N(t) \in V_N$,

$$\int_\Omega (\partial_t u_N)v\,dx + \int_\Omega \langle a(x)\nabla u_N, \nabla v\rangle\,dx = 0 \quad \text{for all } v \in V_N,$$

and

$$u_N(0) = \pi_N(u_0) = \text{orthogonal projection of } u_0 \text{ to } V_N \text{ using the } L^2 \text{ inner product.}$$

(a) Show that $w_N = u_N - \pi_N(u)$ satisfies an estimate of the form

$$\frac{d}{dt}\int_\Omega |w_N|^2\,dx + C_1\int_\Omega |w_N|^2\,dx \le C_2\int_\Omega |\nabla u - \nabla\pi_N(u)|^2\,dx,$$

where C_1 and C_2 are positive constants.

(b) Show that any function $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is well-approximated in H^1 by its L^2 projection to V_N, in the sense that

$$\int_\Omega |\nabla u - \nabla \pi_N(u)|^2 \, dx \leq \frac{1}{\lambda_N} \int_\Omega |\nabla \nabla u|^2 \, dx.$$

(c) When $\partial\Omega$ is nice enough, it is known that $\lambda_N \sim C_\Omega N^{2/n}$. (This is known as Weyl's law. You can find a formula for C_Ω in Section 6.5 of Evans. A proof can be found in volume 1 of Courant & Hilbert's *Methods of Mathematical Physics*, which is available online through Bobcat.) Also: for initial data $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, the PDE solution u remains uniformly bounded for all time in this space. (This is part of the basic existence theory; see e.g. Theorem 5 of Evans' Section 7.1.) Using these facts together with (a) and (b), prove that

$$\|u_N(t) - u(t)\|_{L^2(\Omega)} \leq CN^{-1/n}$$

with a constant C that's independent of time.

(d) Now suppose u is smoother, specifically that $\int_\Omega |\Delta^k u|^2 \, dx$ is uniformly bounded in time for some integer $k \geq 2$. Can you adjust the preceding arguments to get a better estimate for $\|u_N(t) - u(t)\|_{L^2(\Omega)}$ than the one stated in part (c)?

(6) Now let's consider the analogue of Problem 5 for the semilinear heat equation

$$u_t - \Delta u = u^3 \text{ in } \Omega, \text{ with } u = 0 \text{ at } \partial\Omega \text{ and } u = u_0 \text{ at } t = 0,$$

when Ω is a bounded domain in $\mathbb{R}^3$ (with sufficiently nice boundary). We assume that

$$\sup_{0 \leq t \leq T} \int_\Omega |\nabla \nabla u|^2 \, dx \leq M \tag{1}$$

for some constant M. (This amounts to taking $u_0 \in H^2(\Omega)$ and assuming the solution has not blown up by time T.) As in Problem 5, we denote by u_N the solution of the Galerkin approximation obtained using the first N eigenfunctions of the Dirichlet Laplacian; it is determined by the conditions that $u_N(t) \in V_N$ for all t and

$$\int_\Omega (\partial_t u_N) v \, dx + \int_\Omega \langle \nabla u_N, \nabla v \rangle \, dx = \int_\Omega u_N^3 v \, dx \quad \text{for all } v \in V_N,$$

together with the initial condition

$$u_N(t) = \pi_N(u_0)$$

(as before, π_N denotes orthogonal projection from $L^2(\Omega)$ to V_N using the L^2 inner product).

(a) Let $w_N = u_N - \pi_N(u)$. In Problem 5 we relied on an energy estimate involving $\frac{d}{dt} \int_\Omega |w_N|^2 \, dx$, and this problem can be done that way too. However when I sketched the local-in-time existence theory in Lecture 10, I relied mainly on an energy estimate that involves $\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx$, so it is natural in this setting look for a related estimate involving $\frac{d}{dt}\int_\Omega |\nabla w_N|^2\,dx$. Show that in fact

$$\frac{d}{dt}\int_\Omega |\nabla w_N|^2\,dx + 2\int_\Omega |\Delta w_N|^2\,dx = -2\int_\Omega (u_N^3 - u^3)\Delta w_N\,dx.$$

It is convenient to rewrite the integral on the RHS as

$$\int_\Omega (u_N^3 - [\pi_N(u)]^3)\Delta w_N\,dx + \int_\Omega ([\pi_N(u)]^3 - u^3)\Delta w_N\,dx = I + II.$$

(b) Show that

$$I \le C\left(\int_\Omega |\Delta w_N|^2\,dx\right)^{1/2}\left(\int_\Omega w_N^6\,dx\right)^{1/6}\left(\int_\Omega |u_N|^6 + |\pi_N(u)|^6\,dx\right)^{1/3}.$$

As a start toward estimating the last of the three terms in this product, explain why

$$\left(\int_\Omega |\pi_N(u)|^6\,dx\right)^{1/3} \le C\int_\Omega |\nabla \pi_N(u)|^2\,dx \le C\int_\Omega |\nabla u|^2\,dx,$$

and why in combination with (1) this gives

$$\sup_{0\le t\le T}\int_\Omega |\pi_N(u)|^6, dx \le C_1 M.$$

(c) Since we expect to show that u_N is close to $\pi_N(u)$, in light of part (b) it is natural to expect that

$$\sup_{0\le t\le T}\int_\Omega |u_N|^6\,dx \le 2C_1 M. \tag{2}$$

(Estimates proved using this assumption are valid up to the first time when (2) fails. We'll see in part (d) that if N is large enough then it never fails for $t \in [0,T]$.) Argue using (2) and part (b) that for any $\varepsilon > 0$,

$$I \le \varepsilon\int_\Omega |\Delta w_N|^2\,dx + C_{\varepsilon,M}\int_\Omega |\nabla w_N|^2\,dx$$

and

$$II \le \varepsilon\int_\Omega |\Delta w_N|^2\,dx + C_{\varepsilon,M}\int_\Omega |\nabla \pi_N(u) - \nabla u|^2\,dx.$$

Conclude (by arguing as in Problem 5 and using that the spatial dimension is $n = 3$) that

$$\frac{d}{dt}\int_\Omega |\nabla w_N|^2 \le C_2\int_\Omega |\nabla w_N|^2\,dx + C_3 N^{-2/3},$$

and show using this an estimate of the form

$$\|u(t) - u_N(t)\|^2_{H_0^1(\Omega)} \le CN^{-2/3}$$

up to the first time that (2) fails. (The constant C can depend on M and T, but is otherwise independent of u.)

(d) Show finally that (2) holds for all $t \in [0,T]$ if N is sufficiently large, so that

$$\sup_{0\le t\le T}\|u(t) - u_N(t)\|^2_{H_0^1(\Omega)} \le CN^{-2/3}.$$