PDE, Spring 2020, HW4 – corrected 4/4. Distributed 3/24/2020, due 4/7/2020. (Upload your solution using the Assignments tool in NYU Classes. If possible, please provide a single pdf.) *Problem 1 was wrong in the initial version of this hw, so it has been corrected. Problem 2 was not significantly changed, but it now refers to part (c) of problem 1. In Problem 4, the original version asked for an estimate of the H^2 norm of D^2u; in this corrected version the estimate involves the L^2 norm of D^2u. Problem 5 originally had $Lu = f$; in this corrected version the equation is $Lu = 0$. In Problem 7, the original version involved the pde $-\Delta u = b(\nabla u)$; in this corrected version the PDE is $-\Delta u + \mu u = b(\nabla u)$, with μ sufficiently large (the problem asks you to extend what I did in class to the same PDE with a nonzero Dirichlet boundary condition.)*

(1) Regularity theory for a constant-coefficient elliptic operator in $\mathbb{R}^n$ or in a half-space is much easier than the variable-coefficient case. Let's explore this:

  (a) Show using a Fourier-based argument that for $f \in L^2(\mathbb{R}^n)$, there is a unique $u \in H^2(\mathbb{R}^n)$ satisfying $-\Delta u + u = f$, and it satisfies $\|u\|_{H^2} \leq C\|f\|_{L^2}$.

  (b) Show that the analogous assertion for $-\Delta u = f$ is false, by giving an example of $f \in L^2(\mathbb{R})$ such that $-u_{xx} = f$ has no L^2 solution.

  (c) Suppose now that $u \in L^2(\mathbb{R}^n)$ and its distributional Laplacian $-\Delta u = f$ is in L^2. Show that $u \in H^2(\mathbb{R}^n)$ and $\|u\|_{H^2} \leq C(\|u\|_{L^2} + \|f\|_{L^2})$, with a constant C that depends only on n.

  (d) Now consider the halfspace $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \text{ such that } x_n > 0\}$. Use odd reflection and part (c) to show that if $u \in H^1_0(\mathbb{R}^n_+)$ and its distributional Laplacian $-\Delta u = f$ is in $L^2(\mathbb{R}^n_+)$ then $\|u\|_{H^2(\mathbb{R}^n_+)} \leq C(\|u\|_{L^2(\mathbb{R}^n_+)} + \|f\|_{L^2(\mathbb{R}^n_+)})$.

(2) Continuing in the spirit of Problem 1, suppose u solves $-\Delta u = f$ in a domain Ω, and Ω' is strictly smaller in the sense that its closure is contained in Ω. Let ϕ be a smooth, compactly supported function satisfying $\phi = 1$ in Ω' and $\phi = 0$ near $\partial\Omega$, and consider $w = \phi u$.

  (a) Find Δw.

  (b) Use Problem 1(c) (combined with suitable inequalities) to show the interior regularity result $\|u\|_{H^2}(\Omega') \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$.

(3) Our discussion of regularity used some lemmas about difference quotients. This problem asks for analogous results about second-differences. To capture the main idea with a minimum of notational complexity, let's work in one space dimension: the second-difference operator D_h^2 of a function h is defined by

$$D_h^2 u(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{h^2}.$$

  (a) Suppose Ω is an open interval in $\mathbb{R}$ and Ω' is strictly smaller in the sense that its closure is contained in Ω. Show that if $u \in W^{2,p}(\Omega')$ for $1 \leq p < \infty$ then $\|D_h^2 u\|_{L^p(\Omega')} \leq C\|u''\|_{L^p(\Omega}$ with C independent of h, provided h is small enough that $D_h^2 u$ is well-defined in Ω'.

(b) Now assume $1 < p < \infty$, and suppose $u \in L^p(\Omega)$ has the property that $\|D_h^2 u\|_{L^p(\Omega)} \leq C$ for all sufficiently small h (with C independent of h). Show that the second distributional derivative of u exists and is in L^p.

(c) Show, by giving a counterexample, that the assertion of (b) can be false when $p = 1$.

(4) [This is problem 7 in Chapter 6 of Evans.] Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE $-\Delta u + c(u) = f$ in all $\mathbb{R}^n$, where $f \in L^2$ and the function $c : \mathbb{R} \to \mathbb{R}$ is smooth with $c(0) = 0$ and $c' \geq 0$. Assume also that $c(u) \in L^2(\mathbb{R}^n)$. Show that

$$\|D^2 u\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

with a constant C that's independent of u and f. [Hint: argue as we did for interior regularity in the variable-coefficient linear setting, but without a cutoff function.]

(5) [This is problem 8 in Chapter 6 of Evans.] Let u be a smooth solution of $Lu = -\sum_{i,j} a_{ij}(x) D_{ij}^2 u = 0$ in Ω, where the coefficient is elliptic and has bounded derivatives.

(a) Show that $v = |Du|^2 + \lambda u^2$ satisfies $Lv \leq 0$ if λ is large enough.

(b) Deduce that $\|Du\|_{L^\infty(\Omega)} \leq C\left(\|Du\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\right)$.

(6) Let Ω be a bounded domain in $\mathbb{R}^n$, and let $b : \mathbb{R}^n \to \mathbb{R}$ be smooth. Consider the nonlinear boundary value problem

$$-\Delta u = b(\nabla u) \text{ in } \Omega, \text{ with } u = 0 \text{ at } \partial\Omega.$$

Show that there can be at most one smooth solution. [Hint: if there are two solutions, subtract the two equations and use that $b(\eta) - b(\xi) = \left(\int_0^1 \nabla b[\xi + t(\eta - \xi)]\, dt\right) \cdot (\eta - \xi)$.]

(7) In Lecture 8, I applied the Schauder fixed point theorem to prove existence of an H^2 solution of $-\Delta u + \mu u = b(\nabla u)$ in a bounded domain Ω with $u = 0$ at $\partial\Omega$, provided b is globally Lipschitz and μ is large enough. Now consider the same PDE with a nonzero Dirichlet boundary condition

$$-\Delta u + \mu u = b(\nabla u) \text{ in } \Omega, \text{ with } u = g \text{ at } \partial\Omega.$$

Assume there is a harmonic function G with boundary value g and $|\nabla G|$ uniformly bounded. Show that this problem, too, has a solution if b is globally Lipschitz and μ is sufficiently large. [Hint: rewrite the PDE as an equation for $\tilde{u} = u - G$, and use a Schauder-based argument similar in spirit to what I did in Lecture 8.]