PDE, Spring 2020, HW3. Distributed 3/3/2020, due in class 3/24/2020. [You have 3 weeks due to spring break].

(1) Given a function $\phi\colon \mathbb{R}^n \to R$, find a variational problem whose Euler-Lagrange equation equation is

$$-\Delta u + \nabla\phi \cdot \nabla u = f \quad \text{in } \Omega$$

with $u = 0$ at $\partial\Omega$.

(2) Let Ω be a bounded domain in $\mathbb{R}^n$, and let x_0 be a point in Ω. Is the variational principle

$$\inf_{u=0 \,\text{at}\, \partial\Omega} \int_\Omega \frac{1}{2}|\nabla u|^2\, dx - u(x_0)$$

bounded below, as u ranges over smooth functions? (Comment: while we usually consider linear functionals of the form $\int_\Omega uf\, dx$, this one arises by taking f to be a measure – namely $f = \delta_{x_0}$.) Warning: the answer is different for $n = 1$ vs $n \geq 2$.

(3) In some cases, the notion of a "local minimizer" depends on the choice of topology. Consider the 1D variational problem

$$\min_{u(0)=a,\, u(1)=b} \int_0^1 (u_x^2 - 1)^2\, dx$$

with $a < b$ chosen so that $|b-a| < 1$ and $W(\xi) = (\xi^2-1)^2$ has $W''(b-a) > 0$.

(a) Show that the minimum value is 0 by displaying a minimizer. (A convincing picture is sufficient.)

(b) Show that the affine function $u_*(x) = a + (b-a)x$ is a local minimizer in the C^1 topology, in the sense that if $v \in C^1(0,1)$ has $v(0) = a$, $v(1) = b$, and $\|v - u_*\|_{C^1}$ sufficiently small, then $\int_0^1 W(v_x), dx \geq \int_0^1 W(u_{*x})\, dx$. (Hint: start by showing that the function $t \mapsto \int_0^1 W\big(u_{*x} + t(v_x - u_{*x})\big)\, dx$ is convex.)

(c) Show that u_* is not a local minimizer in the L^∞ topology. (Again, a convincing picture is sufficient.)

(4) Our discussion of numerical approximation was limited to variational problems, however the same techniques can also be used for linear PDE's that don't come from variational problems. They work whenever the Lax-Milgram lemma assures existence of a unique solution. In fact, let H be a Hilbert space and let $f : H \to \mathbb{R}$ be a continuous linear functional. Suppose $B(u, v)$ is a bilinear (not necessarily symmetric) functional on $H \times H$ such that

$$|B(u,v)| \leq \alpha\|u\|_H\|v\|_H \quad \text{and} \quad B(u,u) \geq \beta\|u\|_H^2$$

for some positive constants α and β. The Lax-Milgram lemma gives the existence of a unique u_* such that

$$B(u_*, v) = \langle f, v\rangle \quad \text{for all } v \in H.$$

To approximate u_* numerically, it is natural to consider a finite-dimensional subspace $H_N \subset H$, and to seek $u_N \in H_N$ such that

$$B(u_N, v) = \langle f, v \rangle \quad \text{for all } v \in H_N.$$

(a) Show that u_N exists and is unique.

(b) Show that $\|u_* - u_N\|_H \leq \frac{\alpha}{\beta} \min_{v \in H_N} \|u_* - v\|_H$

Problems (5)–(7) consider a variational problem of the form

$$\min_{u=g \text{ at } \partial\Omega} \int_\Omega W(\nabla u) - uf \, dx \tag{1}$$

where $W \colon \mathbb{R}^n \to \mathbb{R}$ is convex with "pth-power growth" for some $p > 1$, in the sense that

$$C_1(|\xi|^p - 1) \leq W(\xi) \leq C_2(|\xi|^p + 1). \tag{2}$$

(Recall that in Lecture 5, we applied the direct method of the calculus of variations to prove existence of a minimizer.)

(5) The formal Euler-Lagrange equation for the variational problem (1) is

$$-\mathrm{div}\,(DW(\nabla u)) = f.$$

Its weak form is the assertion that $u \in W^{1,p}(\Omega)$ has $u|_{\partial\Omega} = 0$ and satisfies

$$\int_\Omega \langle DW(\nabla u), \nabla v \rangle - fv \, dx = 0 \tag{3}$$

for any $v \in W_0^{1,p}(\Omega)$.

(a) Show that the integral in (3) makes sense if

$$|DW(\xi)| \leq C(|\xi|^{p-1} + 1) \tag{4}$$

for some constant C.

(b) Show that if W is convex and satisfies (2) then it also satisfies (4).

(c) Show that if $u \in W^{1,p}(\Omega)$ solves (1) then it does indeed satisfy (3) for every $v \in W_0^{1,p}(\Omega)$. (Note: your task is to justify the formal calculation, which evaluates the variational problem at $u_t(x) = u(x) + tv(x)$ for t near 0 and takes the derivative at $t = 0$.)

(6) My proof of lower semicontinuity used the fact that if a subset S of $W^{1,p}(\Omega)$ is closed under strong convergence and convex then it is also closed under weak convergence. The lower-semicontinuity of $\int_\Omega W(\nabla u) \, dx$ also has a more elementary proof, starting from the fact that if W is convex with pth power growth for $p > 1$ then

$$\int_\Omega W(\nabla u) \, dx = \sup_{\eta \in L^\infty(\Omega, \mathbb{R}^n)} \int_\Omega \langle \nabla u, \eta \rangle - W^*(\eta(x)) \, dx \tag{5}$$

where W^* is the Legendre transform of W, defined by

$$W^*(\eta) = \sup_{\xi \in \mathbb{R}^n} \langle \xi, \eta \rangle - W(\xi).$$

(For a brief introduction to the Legendre transform, see Section 3.3 of Evans. I am not asking you to prove the validity of (5), but you might want to think about that. I wrote "sup" not "max" on purpose in (5), since if ∇u is not in L^∞ then the optimal η is typically also not in L^∞.) *OK, here is the question:* Using (5), give a simple proof that $\int_\Omega W(\nabla u)\, dx$ is lower-semicontinuous under weak convergence in $W^{1,p}(\Omega)$.

(7) My discussion of Galerkin-type numerical methods in Lecture 5 was limited to linear problems (quadratic functionals). However one can do something very similar for (1), provided that $W(\xi)$ is uniformly convex in the sense that

$$\sum_{i,j} \frac{\partial^2 W}{\partial \xi_i \partial \xi_j} \xi_i \xi_j \geq C|\xi|^2 \tag{6}$$

for some $C > 0$.

(a) Show that under this hypothesis,

$$W(\xi) - W(\eta) \geq \langle \nabla W(\eta), \xi - \eta \rangle + \frac{C}{2}|\xi - \eta|^2$$

for any $\xi, \eta \in \mathbb{R}^n$.

(b) Show that if u_* minimizes (1) and u_N is any function in $W^{1,p}(\Omega)$ with boundary trace g, our functional $E[u] = \int_\Omega W(\nabla u) - uf\, dx$ satisfies

$$E[u_N] - E[u_*] \geq \frac{C}{2} \int_\Omega |\nabla(u_N - u_*)|^2\, dx.$$

(Note: if u_N minimizes the functional in a finite-dimensional subspace and u_* is well-approximated by a function in the subspace, then the left hand side will be small. Thus, this problem shows that minimization in a subspace is a good way to solve a variational problem numerically.)