PDE, Spring 2020, HW2. Distributed 2/18/2020, due in class 3/3/2020.

(1) Let Ω be a bounded domain (with nice enough boundary) in $\mathbb{R}^n$, and consider $1 \leq p < n$. Recall that for $p^* = \frac{np}{n-p}$ we have

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Show that the inclusion of $W^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$ is not compact.

(2) When we proved that the inclusion of $W^{1,p}(\Omega)$ in $L^q(\Omega)$ is compact for $q < p^*$, a key step was that when u has compact support and $u_\varepsilon = \eta_\varepsilon * u$ is its convolution with a standard mollifer, $\int |u - u_\varepsilon|\, dx \leq C \int |\nabla u|\, dx$ while $|u|$ and $|\nabla u|$ are controlled pointwise negative powers of ε times $\int |u|\, dx$.

In considering weak solutions of 2nd-order linear elliptic pde, we used just the special case $p = 2, q = 2$, which tells us that the inclusion of $H^1(\Omega)$ in $L^2(\Omega)$ is compact. There is, of course, a Fourier approach to this result. In this setting, a convenient smooth approximation of a function u (with compact support, extended by 0 to all $\mathbb{R}^n$) is the function u_R, defined by

$$\hat{u}_R(\xi) = \begin{cases} \hat{u}(\xi) & \text{for } |\xi| < R \\ 0 & \text{for } |\xi| \geq R. \end{cases}$$

Show that

(a) $\|u - u_R\|_{L^2} \leq \frac{C}{R}\|u\|_{H^1}$, and
(b) for any R, there is a constant $M(R)$ such that $\|u_R\|_{L^\infty} + \|\nabla u_R\|_{L^\infty} \leq M(R)\|u\|_{H^1}$.

[Given this, the proof of the compactness of the inclusion $H^1(\Omega) \subset L^2(\Omega)$ using u_R is entirely parallel to the proof using u_ε. I recommend thinking that through, but I'm not asking you to write it up.]

(3) If Ω is a (sufficiently regular, bounded) domain in $\mathbb{R}^n$, the inclusion of $H^2(\Omega)$ in $L^2(\Omega)$ is compact. (This can be proved using mollification, or using the approach of problem 2.) Using this, show the existence of a constant C (depending on Ω) such that

$$\min_{p \in L} \int_\Omega |u - p|^2\, dx \leq C \int_\Omega |\nabla\nabla u|^2\, dx,$$

where $|\nabla\nabla u|^2 = \sum_{i,j} (\partial^2 u / \partial x_i \partial x_j)^2$ and L is the class of affine functions of x (equivalently: L is the finite-dimensional vector space of functions satisfying $\nabla\nabla u = 0$).

(4) Let $\Omega = [-1, 1] \times [0, 1]$ and suppose $a(x_1, x_2)$ is piecewise constant:

$$a(x_1, x_2) = \begin{cases} a_1 & \text{if } x_1 < 0 \\ a_2 & \text{if } x_1 > 0, \end{cases}$$

where a_1 and a_2 are both positive. Suppose u is smooth except at the line $x_1 = 0$ where $a(x)$ jumps, and u is a weak solution of

$$-\text{div}(a(x)\nabla u) + \frac{\partial u}{\partial x_1} = f$$

in Ω. What jump or continuity conditions must hold for u and its normal derivative $\partial u/\partial x_1$ at $x_1 = 0$?

(5) If K is a self-adjoint compact operator on L^2, the Fredholm alternative tells us that $(I - K)u = f$ has a solution exactly when u is orthogonal to the nullspace of $(I - K)$. Show how this implies that for the PDE $-\Delta u - \lambda u = f$ in a bounded domain Ω with the Dirichlet boundary condition $u = 0$ at $\partial\Omega$,

(a) if λ^{-1} is not an eigenvalue of $(-\Delta_D)^{-1}$ (using the notation of my Lecture 3 notes), then the equation has a unique weak solution; and

(b) if λ^{-1} is an eigenvalue of $(-\Delta_D)^{-1}$, then a solution exists if and only if f is orthogonal (in $L^2(\Omega)$) to the nullspace of $(-\Delta_D)^{-1} - \lambda^{-1}I$.

(6) Let Ω be a bounded domain, and let Γ be a subset of $\partial\Omega$ with positive codimension-one measure. The existence of a constant C such that

$$\int_\Omega |u|^2\,dx \le C \int_\Omega |\nabla u|^2\,dx \quad \text{whenever } u|_\Gamma = 0$$

can be proved by a compactness argument, but this gives no information about the constant C. Give a different argument (not by contradiction), based on the estimate $\int_\Omega |u - \overline{u}|^2\,dx \le C_1 \int_\Omega |\nabla u|^2\,dx$, which gives more explicit control over the value of C (given that you know C_1). (Hint: use the trace theorem, applied to $u - \overline{u}$.)

(7) Specialize our treatment of weak solutions to the case when the Hilbert space is $H^2(\Omega)$,

$$B(u, v) = \int_\Omega \langle \nabla\nabla u, \nabla\nabla v \rangle\,dx,$$

and the linear functional is $L_f(v) = \int_\Omega f v\,dx$ for some $f \in L^2(\Omega)$.

(a) This leads, of course, to weak solutions of $\Delta^2 u = f$. What are the boundary conditions?

(b) Are there consistency conditions on f? Identify them, and show that you haven't missed any.

(c) Does the theory also work for a linear functional of the form $\tilde{L}_f(v) = \int_\Omega f(\partial v/\partial x_1)\,dx$, where $f \in L^2(\Omega)$?

(8) Consider Laplace's equation with a Robin boundary condition:

$$-\Delta u = f \text{ in } \Omega, \text{ with } u + \tfrac{\partial u}{\partial n} = g \text{ at } \partial\Omega.$$

(a) How can our Hilbert-space-based theory of weak solutions be applied in this setting?

(b) What if the boundary condition is changed to $-u + \frac{\partial u}{\partial n} = g$? Can that problem also be handled by our methods?