PDE, Spring 2020, HW1. Distributed 2/5/2020, due in class 2/18/2020.

As you'll probably recognize, many (though not all) these problems are from Evans' book.

(1) Let B be the unit ball in $\mathbb{R}^2$, and let $1 \leq p < \infty$ be fixed. Show that there is no continuous linear map $T : L^p(B) \to L^p(\partial B)$ such that $T(u) = u|_{\partial B}$ when u is continuous on $\overline{B}$. (Briefly: it does not make sense to consider the "boundary trace" of a function in $L^p(B)$. Note: there is nothing special about balls, and nothing special about $\mathbb{R}^2$; I have asked the question in a special case simply to make the answer easy to write down.)

(2) As I mentioned in Lecture 1, the Neumann problem for Laplace's equation in a bounded domain

$$\Delta u = 0 \text{ in } \Omega, \text{ with } \partial u/\partial n = g \text{ at } \partial\Omega$$

cannot be solved by considering the variational problem

$$\min_{\partial u/\partial n = g \text{ at } \partial\Omega} \int_\Omega |\nabla u|^2 \, dx \qquad \text{(WRONG)}.$$

To keep things simple, consider the special case when Ω is the interval $(0, 1)$. What is the minimum value of this (wrong) variational problem? (This question is very closely related to problem 1: it reflects the fact that $u_x(0)$ and $u_x(1)$ are not well-defined, for functions $u \in H^1(0, 1)$.)

(3) Suppose Ω is a connected domain in $\mathbb{R}^n$. Show that if $u \in W^{1,p}(\Omega)$ and $Du = 0$ a.e. then u is constant a.e. (Note: nothing is assumed about the regularity of $\partial\Omega$.)

(4) Consider a function $u \in W^{1,p}(\Omega)$, where Ω is a bounded domain in R^n and $1 \leq p < \infty$.

(a) Show that $|u|$ is in $W^{1,p}(\Omega)$.

(b) Let u^+ and u^- be the positive and negative parts of u (so $u = u^+ - u^-$ and $|u| = u^+ + u^-$). Show that u^+ and u^- are both in $W^{1,p}(\Omega)$ and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\} \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geq 0\} \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$

(Hint: $u^+ = \lim_{\epsilon \to 0} F_\epsilon(u)$, where $F_\epsilon(z) = (z^2 + \epsilon^2)^{1/2} - \epsilon$ for $z \geq 0$ and $F_\epsilon = 0$ for $z < 0$.)

(c) Show that $Du = 0$ a.e. on the set where $u = 0$.

(5) Suppose Ω is a bounded domain with C^1 boundary, and let ξ be a C^1 vector field defined on Ω such that $\xi \cdot n \geq 1$ on $\partial\Omega$.

(a) Assuming $1 \leq p < \infty$, apply the divergence theorem to $\int_{\partial\Omega} |u|^p \, \xi \cdot n \, dS$ to give another proof that when u is smooth,

$$\int_{\partial\Omega} |u|^p \, dx \leq C \int_{\Omega} |Du|^p + |u|^p \, dx.$$

(Remember: our trace theorem – showing that $W^{1,p}$ functions have well-defined boundary traces in L^p of the boundary – followed immediately from this inequality, using the density of smooth functions in $W^{1,p}(\Omega)$.)

(b) Now suppose Ω is a polygon in $\mathbb{R}^2$. Its boundary is not C^1, but it is easy to see that there is nevertheless a C^1 vector field with $\xi \cdot n \geq 1$ at $\partial\Omega$. Does your argument for part (a) still work in this case?

(6) I mentioned in class that for domains with nice enough boundaries, the boundary trace map from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$ is surjective when $p = 1$ but not when $p > 1$. Let's confirm the latter statement for $p = 2$, by proving a sharper estimate on the boundary trace map.

(a) Argue by scaling that we should expect an estimate of the form

$$\left(\int_{\partial\Omega} |u|^q \, dx \right)^{1/q} \leq C \|u\|_{W^{1,2}(\Omega)}$$

when $q \leq 2(n-1)/(n-2)$ and $\Omega \subset \mathbb{R}^n$ with $n > 2$.

(b) Show this statement is correct. (Hint: start by substituting $w = u^q$ into the known estimate $\int_{\partial\Omega} |w| \, dx \leq C \|w\|_{W^{1,1}(\Omega)}$.)

(7) Let g be a smooth function on the unit circle, and let its Fourier series be

$$g(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta).$$

(a) Show that the function

$$u = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta).$$

is harmonic. (It is the unique harmonic function with boundary value g, but I'm not asking you to prove this.)

(b) Show that if B is the unit ball, then

$$\int_B |\nabla u|^2 \, dx = c \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2)$$

for some constant c (independent of g).

(c) Conclude that when B is the unit ball, the exact space of boundary traces of $H^1(B)$ functions is the closure of the smooth functions on the unit circle under the norm $H^{1/2}$ defined by

$$\|g\|^2_{H^{1/2}} = a_0^2 + \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2).$$

(d) Check that the piecewise constant function

$$g(\theta) = \begin{cases} 1 & \text{for } 0 < \theta < \pi \\ -1 & \text{for } \pi < \theta < 2\pi \end{cases}$$

does not have bounded $H^{1/2}$ norm. (It has a perfectly good harmonic extension to the ball, given by the function u in part (a); however u does not have finite Dirichlet norm.)

(8) In $\mathbb{R}^3 = \{(x, y, z)\}$, let L be the line $y = z = 0$. Show that if $s > 1$ then there is a well-defined restriction map $R : H^s(\mathbb{R}^3) \to H^{s-1}(L)$, determined by the property that Ru is the restriction of u to L when u is smooth.

(9) A special case of the Sobolev embedding theorem says that if Ω is a bounded domain in $\mathbb{R}^2$ with a boundary nice enough for the extension lemma to hold, then

$$\|u\|_{L^p(\Omega)} \leq C_p \|u\|_{W^{1,2}(\Omega)} \quad \text{for any } 1 \leq p < \infty.$$

Give an example to show that this estimate can fail in a domain with a sharp enough cusp. (Hint: consider a cusp whose tip is $x = y = 0$, and whose interior has the form $\{(x, y) : |x| < y^m,\ y > 0\}$, and a function that's equal to $y^{-\alpha}$ near the cusp.)