Take-Home Final Exam
PDE, Spring 2020
May 19, 2020

This exam will be available from the Assignments section of the NYU Classes site starting 9am (NY time) on Tuesday, May 19, and it will also be distributed by email at that time. It is intended to take just a couple of hours, but you may spend longer if you wish. Your solutions must be uploaded to the Assignments section of the NYU Classes site by 9am (NY time) on Wednesday, May 20.

This is an open-book, open-note exam: you may use my lecture notes and problem set solutions, your own notes, our textbooks, other books, internet resources, etc. *However you may not consult with other people (classmates or friends or anyone else)*; this includes communication by voice, email, or other means. Please wait until after the exam is over to discuss it with your classmates and friends.

The exam has 8 questions, some with several parts. Each of the 8 questions is worth 10 points (although some may be easier or more time-consuming than others). The maximum score is thus 80.

If you have questions about the exam, send email to kohn@cims.nyu.edu. I will check my email at least every two hours during the period 9am – 9pm (NY time) on Tues May 19. Corrections or clarifications (if any) will be distributed by email.

(1) Let $\phi(x)$ be a smooth, radially symmetric function with compact support such that $\int \phi(x)\,dx = 1$. Reviewing standard notation: in space dimension n we define $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$, and for any $u \in L^1(\mathbb{R}^n)$ we write u_ε for the convolution $\phi_\varepsilon * u$, in other words $u_\varepsilon(x) = \int_{\mathbb{R}^n} \phi_\varepsilon(x-y)u(y)\,dy$. Show that if $\frac{\partial u}{\partial x_j} = f$ in the sense of distributions with $f \in L^1$ then $\frac{\partial u_\varepsilon}{\partial x_j} = f_\varepsilon$.

(2) For each of the following examples, explain why the Lax-Milgram lemma assures the existence of a unique u, and identify the boundary value problem it solves if you accept that u is smooth. In each part, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary.

(a) The function $u \in H^1(\Omega)$ satisfies

$$\int_\Omega \langle \nabla u, \nabla v\rangle + uv\,dx = \int_{\partial\Omega} fv\,dA \quad \text{for every } v \in H^1(\Omega),$$

for some given smooth function $f : \partial\Omega \to \mathbb{R}$. (In the boundary integral, dA represents surface area.)

(b) The function $u \in H^1(\Omega)$ has mean value 0 and satisfies

$$\int_\Omega \langle \nabla u, \nabla v\rangle\,dx = \int_\Omega \frac{\partial v}{\partial x_1}\,dx \quad \text{for every } v \in H^1(\Omega).$$

(Hint toward identifying the PDE: the RHS can be written as a boundary integral.)

(3) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and let x_0 be a point in Ω. Consider the minimization

$$\inf_{\substack{u=0 \text{ at } \partial\Omega \\ u(x_0)=1}} \int_\Omega |\nabla u|^2 \, dx$$

where u ranges over smooth functions.

(a) Show that for $n \geq 2$ the minimum value is 0.

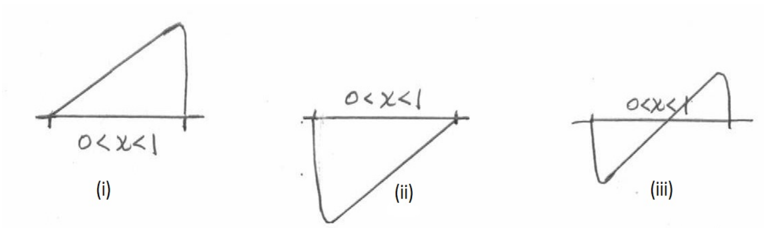
(b) Explain why $n = 1$ is different, and identify the minimum value when $\Omega = (-1, 1)$ and $x_0 = 0$.

(4) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function that's nondecreasing (i.e. $f' \geq 0$). Show that the boundary value problem

$$-\Delta u + f(u) = 0 \text{ in } \Omega, \text{ with } u = \phi \text{ at } \partial\Omega$$

can have at most one classical solution.

(5) Consider the three sketches below, labeled (i)–(iii).



(a) Which of the figures is consistent with u solving the boundary value problem $\varepsilon u_{xx} + u_x = 1$ for $0 < x < 1$, with $u(0) = u(1) = 0$ and $\varepsilon > 0$? (You must briefly justify your answer to get credit.)

(b) Which of the figures is consistent with u solving the boundary value problem $\varepsilon u_{xx} + u_x = 1$ for $0 < x < 1$, with $u(0) = u(1) = 0$ and $\varepsilon < 0$? (Here too, you must briefly justify your answer to get credit.)

(6) Consider the parabolic initial value problem

$$u_t - u_{xx} = uu_x \text{ for } 0 < x < 1,$$
$$u = 0 \text{ at } x = 0 \text{ and } x = 1,$$
$$u(x, 0) = g(x) \text{ at } t = 0.$$

(a) Show that if u is a classical solution then

$$\frac{d}{dt} \int_0^1 u^2 \, dx + 2 \int_0^1 u_x^2 \, dx = 0.$$

(b) Let $v_1, \ldots, v_N$ be smooth functions on $(0,1)$ that vanish at $x = 0$ and $x = 1$, and that are orthogonal in the L^2 norm (i.e. $\int_0^1 v_i v_j \, dx = 0$ for $i \neq j$). Let V be their span, and consider the Galerkin approximation $u^N(x,t) = \sum_j a_j(t) v_j(x)$ characterized by

$$\int_0^1 u_t^N v + u_x^N v_x \, dx = \int_0^1 u^N u_x^N v \, dx \quad \text{for all } v \in V \text{ and all } t > 0$$

together with the initial condition that $u^N(x,0)$ is the orthogonal projection of g onto V using the L^2 norm. Show that u^N exists for all t.

(7) In HW5 we considered a semilinear heat equation in space dimension one. This problem asks how some parts of that calculation extend to space dimension n. Throughout this problem, I write H^k for the space $H^k(\mathbb{R}^n)$.

(a) Show that in any space dimension n and for any nonnegative integer k, $e^{t\Delta}$ is a bounded linear map from H^k to itself, with operator norm at most 1 (in other words, $\|e^{t\Delta} u\|_{H^k} \leq \|u\|_{H^k}$).

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(0) = 0$. Show that if $u \in C(0,T;H^k)$ with $k > n/2$ and

$$\sup_{0 \leq t \leq T} \|u\|_{H^k} \leq M,$$

then there is a constant C (depending only on k, n, f, and M) such that

$$\left\| \int_0^t e^{(t-s)\Delta} f(u(s)) \, ds \right\|_{H^k} \leq Ct \quad \text{for any } t,\ 0 \leq t \leq T.$$

(c) Now show that if u_1 and u_2 are both in $C(0,T;H^k)$ with $k > n/2$ and

$$\sup_{0 \leq t \leq T} \|u_j\|_{H^k} \leq M \quad \text{for } j = 1, 2,$$

then there is a constant C' (depending only on k, n, f, and M) such that

$$\left\| \int_0^t e^{(t-s)\Delta} (f(u_1(s)) - f(u_2(s))) \, ds \right\|_{L^2} \leq C' t \sup_{0 \leq t \leq T} \|u_1(s) - u_2(s)\|_{L^2}$$

for any t, $0 \leq t \leq T$.

(8) Recall that we say u is a viscosity solution of $H(\nabla u) = 0$ in a region Ω if

- whenever $u - \phi$ has a local max at $x_0 \in \Omega$ with $\phi \in C^2$, $H(\nabla \phi(x_0)) \leq 0$, and
- whenever $u - \phi$ has a local min at $x_0 \in \Omega$ with $\phi \in C^2$, $H(\nabla \phi(x_0)) \geq 0$.

(a) Show that a classical (C^2) solution of $H(\nabla u) = 0$ is also a viscosity solution.

(b) Show that if u is a viscosity solution of $H(\nabla u) = 0$ and it is also C^2, then u solves $H(\nabla u)$ in the classical (pointwise) sense.