Intro to the paper "Dynamic trading with predictable returns and transaction costs," N. Garleanu & L.H. Pedersen, J Finance 68 (2013) pp 2309-2340

Goal: use discrete-time dynamic programming to consider how transaction costs should influence investment decisions.

The paper discusses a model with many stocks, examining how portfolio should be changed as time evolves (a time-dependent, transaction-cost-sensitive version of the Markowitz portfolio optimization problem).

I'll discuss just the simplest special case (Example 1 in the paper), when there is just a single stock.

\[ x_t = \text{investor's position in this stock at time } t \] (known for \( t=-1 \), to be decided today \( t=0 \), and eventually at future times \( 1, 2, 3, \ldots \))

and, at each time, investor has an estimate of the stock’s anticipated return. Actually, we will focus only on excess returns \( r_{t+1} = p_{t+1} - (1+r)^T p_t \) where \( p_t \) is stock price at the \( t \); we suppose
Investor knows \( f_t \) at time \( t \), where

\[
f_{t+1} = f_t + u_{t+1}
\]

\( u_{t+1} \) mean zero, independent at each \( t \), variance \( \sigma^2 \).

We assume mean reverting dynamics for \( f_t \):

\[
f_{t+1} - f_t = -\alpha f_t + \varepsilon_{t+1}
\]

\( \varepsilon_{t+1} \) mean zero, independent at each \( t \), variance \( \sigma^2 \).

We assume quadratic transaction costs: a trade of size \( \Delta x \) incurs transaction costs \( \frac{1}{2} \lambda (\Delta x)^2 \) for some constant \( \lambda > 0 \). (Logic of this: it assumes the trade moves the market transiently by an amount that's linear in \( \Delta x \).)

Use discount factor \( 1 - \rho \) in discounting future income.

Investor's problem: at time 0, starting with position \( x \) of stock, knowing \( f_t \) for every period's return, choose \( x_0 \) to maximize

\[\text{risk-adjusted return in 1st period less trans. cost plus expected value of future periods.}\]
Call this \( V(x_1, f_0) \). Principle of dyn. prog. tells us (with \( \Delta x_t = x_t - x_{t+1} \))

\[
V(x_1, f_0) = \max_{x_0} \left\{ -\frac{1}{2} (\Delta x_0)^2 + (1-q) (f_0 - \frac{1}{2} \sigma^2 x_0^2) \right\}
\]

This is a stochastic "LQR-type" problem, so it's natural to guess that \( V \) has the form

\[
V(x_t, f_{t+1}) = -\frac{1}{2} A_{xx} x_t^2 + A_{xf} x_t f_{t+1} + \frac{1}{2} A_{ff} f_{t+1}^2 + A_0
\]

for some constants \( A_{xx}, A_{xf}, A_{ff}, A_0 \).

To find these constants + the optimal investment policy, we substitute this hypothesis into the PDE of dyn. programming:

\[
\text{LHS} = -\frac{1}{2} A_{xx} x_1^2 + A_{xf} x_1 f_0 + \frac{1}{2} A_{ff} f_0^2 + A_0
\]
\[ \text{RHS} = \max \text{ wrt } x_0 \text{ of } \]
\[-\frac{1}{2} (x_0 - x_f)^2 \Lambda + (1 - \rho) \left( x_0 s_0 - \frac{\rho}{2} \sigma^2 x_0^2 \right) + (1 - \rho) \left[ -\frac{1}{2} A_{xx} x_0^2 + A_{xf} x_0 s_0 (1 - \rho) + A_{ff} s_0^2 (1 - \rho) \right] \]

We observe that RHS is quadratic poly in \( x_0 \)

\[-\frac{1}{2} x_0^2 \left( \Lambda + (1 - \rho) \sigma^2 + (1 - \rho) A_{xx} \right) + x_0 \left( \Lambda x_f + (1 - \rho) s_0 (1 - \rho) A_{xf} s_0 \right) + \left( -\frac{1}{2} x_f^2 \Lambda + (1 - \rho) A_{ff} s_0^2 (1 - \rho)^2 + (1 - \rho) A_{ff} \right) \]

Writing this as \(-\frac{1}{2} x_0^2 p + x_0 q + r\) we see that \( x_0 = p/q \) and

\[ \text{RHS} = \frac{1}{2} \frac{q^2}{p} + r = \text{quadratic form in } x_f + s_0 \text{ plus constant} \]

and writing LHS to RHS, determines the values of \( A_{xx}, A_{xf}, A_{ff} \). (Explicit formulas exist - see the article - but I won't try to give them here).

How to understand optimal position \( x_0 \)?

Return to principle of dynamic programming.
\[-\frac{1}{2} A_{xx} x^2 + A_{xf} x f_0 + \frac{1}{2} A_{ff} f^2 + A_0 = \max_x \left\{ -\frac{1}{2} (A x)^T \Lambda + \text{stuff} \right\} \]

This holds for all \( x + f_0 \), so we can differentiate w.r.t. \( x \). Best \( x_0 \) depends on \( x_1 \), but we can use chain rule on RHS:

\[
\frac{d}{dx_1} \text{RHS}(x_1, x_0(x_1)) = \frac{\partial \text{RHS}}{\partial x_1}(x_1, x_0(x_1))
\]

since \( x_0(x_1) \) optimizes RHS w.r.t. \( x_0 \). Thus

\[
(*) \quad -A_{xx} x_1 + A_{xf} f_0 = -(x_1 - x_0) \Lambda
\]

How to interpret this? Let

\[
x_\ast \text{ minimize } -\frac{1}{2} A_{xx} x^2 + A_{xf} x f_0 + \frac{1}{2} A_{ff} f^2 + A_0
\]

i.e.

\[
x_\ast = \frac{A_{xf} f_0}{A_{xx}}
\]

At first you might expect \( x_0 = x_\ast \); but that can't be right, since if \( x_1 \) is far from \( x_\ast \) you'd incur large trans costs to do that trade. Instead, eqn (4) says (after algebraic rearrangement)

\[
x_0 = x_1 (1 - \frac{A_{xx}}{\Lambda}) + \frac{A_{xx}}{\Lambda} x_\ast
\]

Thus: though the "target amount" is \( x_\ast \), due to trans costs you don't go all the way
there — instead you go to a choice just part way between $x_1$ and $x_2$.

\[ x_1 \rightarrow x_2 \]

linear interpolant, with

weights $1 - \frac{x_2}{\lambda}$ and $\frac{x_2}{\lambda}$

(Note: one can see, using the explicit formula for $\Lambda_{xx}$, that $0 < \frac{x_2}{\lambda} < 1$.)

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Rmk: In our HW plan on LQR, we had to consider an ode in time. There is no much ode here because we took the true horizon to be $+\infty$, so the value fn depends only on $x$ (two real variables). This is, of course, what makes the problem amenable to by-hand analysis.