PDE for Finance, Spring 2015 – Homework 6
Distributed 4/27/15, due 5/11/15.

About our final exam: Our Final Exam will be *Monday May 18*, at our usual class time (5:10-7pm), in our usual location (WWH 512). You may bring two 8.5×11 sheets of notes (both sides, any font); the preparation such notes is an excellent study tool. The exam covers the material in Sections 1-7 and Homeworks 1-6; in particular, material covered in class on 5/4 or 5/11 will not be on the exam. For an idea what to expect, see problems 1-6 of my 2003 exam, at http://www.math.nyu.edu/faculty/kohn/pde.finance/2003/exam.pdf (problem 7 of the 2003 exam is about jump diffusions, a topic we haven't covered yet this semester.) In general: the exam problems will address topics you have seen on HW or in class, formulated in such a way that if you understand the material each question can be answered relatively quickly.

Plan for the last few lectures: On 4/27 we'll discuss Example 2 from the Section 7 notes, then we'll review why option prices are given by risk-neutral discounted payoffs. On 5/4 we'll discuss an alternative approach to Merton-type portfolio optimization problems, using what is known as the "martingale method." On 5/11 we'll discuss the analogue of the forward and backward Kolmogorov equations, when the asset price is a jump-diffusion.

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This homework set reinforces our discussions of discrete-time dynamic programming problems.

(1) In Example 1 of the Section 7 notes we considered a problem of optimal execution, using a model in which the impact of a trade on the asset price is linear and permanent. In Example 3 of the Section 7 notes we considered a problem of optimal investment, using a model in which the impact of a trade on the asset price is temporary. This problem considers revisits the optimal execution problem of Example 1, using the impact model of Example 3.

As in Example 1, we consider an investor who wants to buy a fixed amount of a given asset. Let P_i be the price of the asset on day i, ignoring any impact of the investor's trades. (We assume that the impact of his trading within day i is so transient that it does not affect the asset price on day $i+1$). We assume that

$$P_i = P_{i-1} + \sigma e_i$$

where e_i has mean value 0. (The variance and other statistical properties of e_i will not matter to our model.) Concerning market impact: we assume that if the investor buys s units of the asset on day i and yesterday's asset price was P_{i-1}, then his expected cost is $E[P_i(s + \frac{1}{2}\theta s^2)] = P_{i-1}(s + \frac{1}{2}\theta s^2)$. Here $\theta > 0$ is a parameter (fixed over time) that captures the effect of market impact upon the investor's cost.

Following the notation of Example 1, let N be the day when the purchase must be completed, and for $i = N, N-1, N-2, \ldots$ let $V_i(P_{i-1}, W_i)$ be the investor's optimal expected cost starting on day i, if the total to be purchased is W_i and the day $i-1$ price was P_{i-1}.

(a) Show that $V_N(P,W) = P(W + \frac{1}{2}\theta W^2)$.

(b) Show that $V_{N-1}(P,W) = P(W + \frac{1}{2}\frac{\theta}{2}W^2)$.

(c) Use the principle of dynamic programming to relate V_{i-1} to V_i, then use an inductive argument to show that $V_i(P,W) = P(W + \frac{1}{2}\theta\frac{1}{N-i+1}W^2)$ for all $i < N$.

(d) What trading strategy does this model suggest?

(2) Consider the following discrete-time analogue of the Merton problem. An investor starts with wealth W_0 at time t_0. He can invest it only in two assets: one is risk-free and earns no interest; the other is risky, with return R_j in the time period from t_j to t_{j+1} (i.e. its price P_j at time t_j satisfies $P_{j+1} = R_j P_j$). Assume that the returns R_j are independent and identically distributed, and that $R_j > 0$ with probability 1. The investor's goal is to maximize $E[h(W_N)]$, where h is a concave utility function. His trades must be self-financing, and no borrowing or short-selling is permitted.

(a) Let W_j be the investor's wealth at time t_j, and let $0 \le \theta_j \le 1$ be the fraction of his wealth invested in the risky asset during the time interval (t_j, t_{j+1}). Give a formula for W_{j+1} in terms of W_j, θ_j, and R_j.

(b) Let $u_j(W)$ be the investor's optimal expected utility of time-t_N wealth, if his wealth at time t_j is W. How does the principle of dynamic programming determine the function $u_j(W)$ in terms of $u_{j+1}(W)$?

(c) Suppose now that the utility function is $h(w) = w^\gamma$ with $0 < \gamma < 1$. Show that $u_{N-1}(W) = gW^\gamma$, where g is a suitable constant. Your answer should include a characterization of g.

(d) Show that in fact

$$u_{N-j}(W) = g^j W^\gamma \quad \text{for } 1 \le j \le N$$

where g is the same constant as in part (c). (Hint: argue inductively.)

(3) Example 2 of the Section 7 notes discusses work by Bertsimas, Kogan, and Lo involving least-square replication of a European option. The analysis there assumes all trades are *self-financing*, so the value of the portfolio at consecutive times is related by

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}).$$

Let's consider what happens if trades are permitted to be non-self-financing. This means we introduce an additional control g_j, the amount of cash added to (if $g_j > 0$) or removed from (if $g_j < 0$) the portfolio at time j, and the portfolio values now satisfy

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}) + g_{j-1}.$$

It is natural to add a quadratic expression involving the g's to the objective: now we seek to minimize

$$E\left[(V_N - F(P_N))^2 + \alpha \sum_{j=0}^{N-1} g_j^2\right]$$

where α is a positive constant. The associated value function is

$$J_i(V, P) = \min_{\theta_i, g_i; \ldots; \theta_{N-1}, g_{N-1}} E_{V_i = V,\, P_i = P} \left[(V_N - F(P_N))^2 + \alpha \sum_{j=i}^{N-1} g_j^2 \right].$$

The claim enunciated in the Section 7 notes remains true in this modified setting: J_i can be expressed as a quadratic polynomial

$$J_i(V_i, P_i) = \bar{a}_i(P_i)|V_i - \bar{b}_i(P_i)|^2 + \bar{c}_i(P_i)$$

where $\bar{a}_i, \bar{b}_i$, and $\bar{c}_i$ are suitably-defined functions which can be constructed inductively. Demonstrate this assertion in the special case $i = N - 1$, and explain how $\bar{a}_{N-1}, \bar{b}_{N-1}, \bar{c}_{N-1}$ are related to the functions $a_{N-1}, b_{N-1}, c_{N-1}$ of the Section 7 notes.