

**PDE for Finance, Spring 2015 – Homework 5**  
**Distributed 4/14/15, due 4/27/15.**

Problem 1 is a classic example (due to Merton) of optimal asset allocation. Problems 2-4 reinforce our discussion of optimal stopping and American options.

1) Consider the following asset-allocation problem. Two investment opportunities are available. One is risk-free, earning (constant) interest  $r$ . The other is lognormal, with (constant) drift  $\mu$  and volatility  $\sigma$ , i.e. it satisfies  $dp = \mu p ds + \sigma p dw$ . You start at time  $t$  by investing wealth  $x$ . Your control is the weighting of your portfolio between these two assets, i.e.

$$\alpha(s) = \text{fraction of wealth invested in the risky asset at time } s$$

subject to  $0 \leq \alpha \leq 1$ . You never withdraw from or add to the portfolio, and you have a fixed horizon  $T$ . Your goal is to maximize the utility of your portfolio value at time  $T$ ; in other words, your value function is

$$u(x, t) = \max_{\alpha(s)} E_{y(t)=x} [h(y(T))]$$

where  $y(s)$  is the value of the portfolio at time  $s$ .

- (a) Find the HJB equation satisfied by  $u$ .
- (b) Find the solution – and the optimal investment strategy – if your utility is  $h(y) = y^\gamma$  with  $0 < \gamma < 1$ .
- (c) Find the solution – and the optimal investment strategy – if your utility is  $h(y) = \log y$ .

2) Example 2 of the Section 6 notes discusses when to sell a stock. The goal proposed in the notes was to maximize the discounted wealth realized by the sale, i.e.

$$\max_{\tau} E_{y(0)=x} [e^{-r\tau} (x - a)]$$

A different goal would be to maximize the discounted *utility* of wealth realized by the sale, i.e.

$$\max_{\tau} E_{y(0)=x} [e^{-r\tau} h(x - a)]$$

where  $h$  is your utility.

- (a) Consider the utility  $h(y) = y^\gamma$  with  $0 < \gamma < 1$ . (This is concave only for  $y > 0$ , but that's OK – it would clearly be foolish to sell at a price that realizes a loss.) Find the value function and the optimal strategy, by arguing as in Example 2 of the notes.
- (b) Show, using a verification argument, that no bounded stopping time does better than the strategy you found in part (a). (Your argument should be parallel to one the notes give for Example 1. I have asked about bounded stopping times so Dynkin's Lemma applies. The assertion can be extended to unbounded stopping times by the method used in the notes for Example 1, but I'm not asking you to do that.)

(c) Example 2 of the notes corresponds to  $\gamma = 1$ . Using  $\gamma < 1$  corresponds to introducing risk-averseness, and decreasing  $\gamma$  corresponds to increasing the risk-averseness. How is this reflected in the  $\gamma$ -dependence of the optimal strategy?

3) In Example 2 of the Section 6 notes we assumed  $\mu < r$ . Let's explore what happens when  $\mu \geq r$ . All other conventions of Example 2 remain in effect: the asset price satisfies  $dy = \mu y dt + \sigma y dw$  and the value function is  $u(x) = \max_{\tau} E_{y(0)=x} [e^{-r\tau} (y(\tau) - a)]$ .

(a) Show that if  $\mu > r$  then  $u = \infty$ .

(b) Show that if  $\mu = r$  then  $u(x) = x$ .

4) For a lognormal underlying with continuous dividend yield  $d$ , the risk-neutral process is  $dy = (r - d)y dt + \sigma y dw$ . The value of a perpetual American call with strike  $K$  is thus

$$u(x) = \max_{\tau} E_{y(0)=x} [e^{-r\tau} (y(\tau) - K)_+]$$

where  $r$  is the risk-free rate.

(a) How is this problem related to Example 2 of the Section 6 notes?

(b) Find the value of this option, and the optimal exercise rule, for  $d > 0$ .

(c) Show that as  $d \rightarrow 0$  the value approaches  $u(x) = x$ .