1. We showed, in the Section 2 notes, that the solution of
\[ w_t = w_{xx} \quad \text{for } t > 0 \text{ and } x > 0, \] with \( w = 0 \) at \( t = 0 \) and \( w = \phi \) at \( x = 0 \) is
\[ w(x, t) = \int_0^t \frac{\partial G}{\partial y}(x, 0, t - s) \phi(s) \, ds \] (1)
where \( G(x, y, s) \) is the probability that a random walker, starting at \( x \) at time 0, reaches \( y \) at time \( s \) without first hitting the barrier at 0. (Here the random walker solves \( dy = \sqrt{2} dw \), i.e. it executes the scaled Brownian whose backward Kolmogorov equation is \( u_t + u_{xx} = 0 \).) Let’s give an alternative demonstration of this fact, following the line of reasoning at the end of the Section 1 notes.

(a) Express, in terms of \( G \), the probability that the random walker (starting at \( x \) at time 0) hits the barrier before time \( t \). Differentiate in \( t \) to get the probability that it hits the barrier at time \( t \). (This is known as the first passage time density).

(b) Use the forward Kolmogorov equation and integration by parts to show that the first passage time density is \( \frac{\partial G}{\partial y}(x, 0, t) \).

(c) Deduce the formula (1).

2. As noted in HW2 problem 5, questions about Brownian motion with drift can often be answered using the Cameron-Martin-Girsanov theorem. But we can also study this process directly. Let’s do so now, for the process \( dz = \mu dt + dw \) with an absorbing barrier at \( z = 0 \).

(a) Suppose the process starts at \( z_0 > 0 \) at time 0. Let \( G(z_0, z, t) \) be the probability that the random walker is at position \( z \) at time \( t \) (and has not yet hit the barrier). Show that
\[ G(z_0, z, t) = \frac{1}{\sqrt{2\pi t}} e^{-|z-z_0|\mu / 2} - \frac{1}{\sqrt{2\pi t}} e^{-\mu z_0} e^{-|z+z_0-\mu t| / 2}. \]
(Hint: just check that this \( G \) solves the relevant forward Kolmogorov equation, with the appropriate boundary and initial conditions.)

(b) Show that the first passage time density is
\[ \frac{1}{2} \frac{\partial G}{\partial \mu}(z_0, 0, t) = \frac{z_0}{t \sqrt{2\pi t}} e^{-|z_0+\mu t| / 2}. \]

3. Consider the linear heat equation \( u_t - u_{xx} = 0 \) on the interval \( 0 < x < 1 \), with boundary condition \( u = 0 \) at \( x = 0, 1 \) and initial condition \( u = 1 \).

(a) Interpret \( u \) as the value of a suitable double-barrier option.
(b) Express \( u(t, x) \) as a Fourier sine series, as explained in Section 3.

(c) At time \( t = 1/100 \), how many terms of the series are required to give \( u(t, x) \) within one percent accuracy?

4. Consider the SDE \( dy = f(y)dt + g(y)dw \). Let \( G(x, y, t) \) be the fundamental solution of the forward Kolmogorov PDE, i.e. the probability that a walker starting at \( x \) at time 0 is at \( y \) at time \( t \). Show that if the infinitesimal generator is self-adjoint, i.e.

\[
-(fu)_x + \frac{1}{2}(g^2u)_{xx} = fu + \frac{1}{2}g^2u_{xx},
\]

then the fundamental solution is symmetric, i.e. \( G(x, y, t) = G(y, x, t) \).

5. Consider the stochastic differential equation \( dy = f(y, s)ds + g(y, s)dw \), and the associated backward and forward Kolmogorov equations

\[
\begin{align*}
  u_t + f(x, t)u_x + \frac{1}{2}g^2(x, t)u_{xx} &= 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T \\
  \rho_s + (f(z, s)\rho)_z - \frac{1}{2}(g^2(z, s)\rho)_{zz} &= 0 \quad \text{for } s > 0, \text{ with } \rho(z) = \rho_0(z) \text{ at } s = 0.
\end{align*}
\]

Recall that \( u(x, t) \) is the expected value (starting from \( x \) at time \( t \)) of payoff \( \Phi(y(T)) \), whereas \( \rho(z, s) \) is the probability distribution of the diffusing state \( y(s) \) (if the initial distribution is \( \rho_0 \)).

(a) The solution of the backward equation has the following property: if \( m = \min_z \Phi(z) \) and \( M = \max_z \Phi(z) \) then \( m \leq u(x, t) \leq M \) for all \( t < T \). Give two distinct justifications:

(a1) Explain why this is an easy consequence of the probabilistic interpretation of \( u \).

(a2) Explain why this amounts to a “maximum principle” for solutions of \( u_t + f(x, t)u_x + \frac{1}{2}g^2(x, t)u_{xx} = 0 \). Then show, by a PDE argument (similar to what we did for \( u_t - u_{xx} = 0 \) in all space, but easier), that such a max principle is valid provided \( |f| \) is uniformly bounded and we know in advance that \( u \) is uniformly bounded. (Hint: let \( \psi \) be a smooth function such that \( \psi(x) = |x| \) for \( |x| \geq 1 \). Consider \( u_{\epsilon, \delta} = u(x, t) \pm \epsilon t \pm \delta \psi \). Apply the maximum principle to \( u_{\epsilon, \delta} \) then consider a suitable limit in which \( \epsilon \) and \( \delta \) tend to 0.)

(b) The solution of the forward equation does not in general have the same property; in particular, \( \max_z \rho(z, s) \) can be larger than the maximum of \( \rho_0 \). Explain why not, by considering the example \( dy = -yds \). (Intuition: \( y(s) \) moves toward the origin; in fact, \( y(s) = e^{-s}y_0 \). Viewing \( y(s) \) as the position of a moving particle, we see that particles tend to collect at the origin no matter where they start. So \( \rho(z, s) \) should be increasingly concentrated at \( z = 0 \).) Show that the solution in this case is \( \rho(z, s) = e^{s}\rho_0(e^{-s}z) \). This counterexample has \( g = 0 \); can you also give a counterexample using \( dy = -yds + \epsilon dw \)?
6. The solution of the forward Kolmogorov equation is a probability density, so we expect
it to be nonnegative (assuming the initial condition \( \rho_0(z) \) is everywhere nonnegative).
In light of Problem 2b it’s natural to worry whether the PDE has this property. Let’s
show that it does.

(a) Consider the initial-boundary-value problem

\[
 w_t = a(x,t)w_{xx} + b(x,t)w_x + c(x,t)w
\]

with \( x \) in the interval \((0, 1)\) and \( 0 < t < T \). We assume as usual that \( a(x,t) > 0 \).
Suppose furthermore that \( c < 0 \) for all \( x \) and \( t \). Show that if \( 0 \leq w \leq M \) at the
initial time and the spatial boundary then \( 0 \leq w \leq M \) for all \( x \) and \( t \). (Hint: a
positive maximum cannot be achieved in the interior or at the final boundary.
Neither can a negative minimum.)

(b) Now consider the same PDE but with \( \max_{x,t} c(x,t) \) positive. Suppose the initial
and boundary data are nonnegative. Show that the solution \( w \) is nonnegative
for all \( x \) and \( t \). (Hint: apply part (a) not to \( w \) but rather to \( \overline{w} = e^{-Ct}w \)
with a suitable choice of \( C \).)

(c) Consider the solution of the forward Kolmogorov equation in the interval, with
\( \rho = 0 \) at the boundary. (It represents the probability of arriving at \( z \) at time \( s \)
without hitting the boundary first.) Show using part (b) that \( \rho(z,s) \geq 0 \) for all
\( s \) and \( z \).

[Comment: statements analogous to (a)-(c) are valid for the initial-value problem
as well, when we solve for all \( x \in \mathbb{R} \) rather than for \( x \) in a bounded domain. The
justification takes a little extra work however, and it requires some hypothesis on the
growth of the solution at \( \infty \).]

7. Consider the solution of

\[
 u_t + au_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T
\]

where \( a \) is a positive constant. Recall that in the stochastic interpretation, \( a \) is
\( \frac{1}{2}g^2 \) where \( g \) represents volatility. Let’s use the maximum principle to understand
qualitatively how the solution depends on volatility.

(a) Show that if \( \Phi_{xx} \geq 0 \) for all \( x \) then \( u_{xx} \geq 0 \) for all \( x \) and \( t \). (Hint: differentiate
the PDE.)

(b) Suppose \( \overline{u} \) solves the analogous equation with \( a \) replaced by \( \overline{a} > a \), using the
same final-time data \( \Phi \). We continue to assume that \( \Phi_{xx} \geq 0 \). Show that \( \overline{u} \geq u \)
for all \( x \) and \( t \). (Hint: \( w = \overline{u} - u \) solves \( w_t + \overline{a}w_{xx} = f \) with \( f = (a - \overline{a})u_{xx} \leq 0 \).)