PDE for Finance Notes, Spring 2003 – Addendum to Section 3.
Notes by Robert V. Kohn, Courant Institute of Mathematical Sciences. For use in connection with the NYU course PDE for Finance, G63.2706.

Numerical solution by finite differences. Before leaving the linear heat equation, let's briefly discuss how it can be solved numerically. These notes consider only the most basic numerical scheme: explicit finite differences, following roughly the discussion in F. John's book. For more information (including more sophisticated schemes) see e.g. Chapter 8 of the "student guide" by Wilmott, Howison, and Dewynne.

But first, some corrections to recent handouts:

- Problem 5b on HW2 was wrong by a factor of 2: the first passage time density is $\frac{1}{2}\frac{\partial G}{\partial z}(z_0, 0, t)$. The extra factor of $\frac{1}{2}$ arises because the process under consideration is Brownian motion with drift, not $\sqrt{2}$ times Brownian motion with drift. The HW2 solution sheet also missed this point. (The solution sheet now posted on my web page has been corrected.)

- When the infinitesimal generator is self-adjoint, the Green's function is symmetric, i.e. $G(x, y, t) = G(y, x, t)$ (see Problem 2 on HW3). Otherwise it isn't (Brownian motion with drift is an example where G is not symmetric – as we know from HW2). So it is a good idea to maintain the convention that $G(x, y, t)$ is the probability of that a walker starting from x at time 0 arrives at y at time t. Section 3 was sloppy about this, in Eqn. (5) and the text just following it. I should have said "It is natural to seek a solution formula in the form

  $$u(y,t) = \int_0^1 G(x, y, t) g(x)\, dy$$

  since $G(x, y, t)$ is then the probability that a random walker starting at x arrives at y at time t without first hitting the boundary."

****

Numerical time-stepping is perhaps the most straightforward method for solving a linear parabolic PDE in a bounded domain. An in-depth treatment is beyond the scope of the present course. But let's spend a moment on the most basic example: an explicit finite-difference scheme for the linear heat equation $f_t = f_{xx}$ on the unit interval $0 < x < 1$ as our spatial interval. We suppose, as usual, that the value of f is specified at the boundary points $x = 0$ and $x = 1$.

If the timestep is Δt and the spatial length scale is Δx then the numerical f is defined at $(x, t) = (j\Delta x, k\Delta t)$. The explicit finite difference scheme determines f at time $(j + 1)\Delta t$ given f at time $j\Delta t$ by reading it off from

$$\frac{f((j+1)\Delta t, k\Delta x) - f(j\Delta t, k\Delta x)}{\Delta t} = \frac{f(j\Delta t, (k+1)\Delta x) - 2f(j\Delta t, k\Delta x) + f(j\Delta t, (k-1)\Delta x)}{(\Delta x)^2}.$$

Notice that we use the initial data to get started, and we use the boundary data when $k\Delta x$ is next to the boundary.

This method has the stability restriction

$$\Delta t < \frac{1}{2}(\Delta x)^2. \tag{1}$$

To see why, observe that the numerical scheme can be rewritten as

$$f((j+1)\Delta t, k\Delta x) = \tfrac{\Delta t}{(\Delta x)^2} f(j\Delta t, (k+1)\Delta x) + \tfrac{\Delta t}{(\Delta x)^2} f(j\Delta t, (k-1)\Delta x) + (1 - 2\tfrac{\Delta t}{(\Delta x)^2}) f(j\Delta t, k\Delta x).$$

If $1 - 2\frac{\Delta t}{(\Delta x)^2} > 0$ then the scheme has a discrete maximum principle: if $f \leq C$ initially and at the boundary then $f \leq C$ for all time; similarly if $f \geq C$ initially and at the boundary then $f \geq C$ for all time. The proof is easy, arguing inductively one timestep at a time. (If the stability restriction is violated then the scheme is unstable, and the discrete solution can grow exponentially.)

One can use this numerical scheme to prove existence (see e.g. John). But let's be less ambitious: let's just show that the numerical solution converges to the solution of the PDE as Δx and Δt tend to 0 while obeying the stability restriction (1). The main point is that the scheme is consistent, i.e.

$$\frac{g(t + \Delta t, x) - g(t, x)}{\Delta t} \to g_t \quad \text{as } \Delta t \to 0$$

and

$$\frac{g(t, x + \Delta x) - 2g(t, x) + g(t, x - \Delta x)}{(\Delta x)^2} \to g_{xx} \quad \text{as } \Delta x \to 0$$

if g is smooth enough. Let f be the numerical solution, g the PDE solution, and consider $h = f - g$ evaluated at gridpoints. Consistency gives

$$\begin{aligned} h((j+1)\Delta t, k\Delta x) &= \tfrac{\Delta t}{(\Delta x)^2} h(j\Delta t, (k+1)\Delta x) + \tfrac{\Delta t}{(\Delta x)^2} h(j\Delta t, (k-1)\Delta x) \\ &\quad + (1 - 2\tfrac{\Delta t}{(\Delta x)^2}) h(j\Delta t, k\Delta x) + \Delta t e(j\Delta t, k\Delta x) \end{aligned}$$

with $|e|$ uniformly small as Δx and Δt tend to zero. Stability – together with the fact that $h = 0$ initially and at the spatial boundary – gives

$$|h(j\Delta t, k\Delta x)| \leq j\Delta t \max |e|.$$

It follows that $h(t, x) \to 0$, uniformly for bounded $t = j\Delta t$, as Δt and Δx tend to 0.

The preceding argument captures, in this special case, a general fact about numerical schemes: that stability plus consistency implies convergence.