PDE for Finance, Spring 2003 – Homework 6
Distributed 4/21/03, due 5/5/03. *Includes a hint for problem 4.*

1) This problem develops a continuous-time analogue of the simple Bertsimas & Lo model of "Optimal control of execution costs" presented in the Section 7 notes. The state is (w, p), where w is the number of shares yet to be purchased and p is the current price per share. The control $\alpha(s)$ is the rate at which shares are purchased. The state equation is:

$$\begin{aligned} dw &= -\alpha\, ds \text{ for } t < s < T, \quad w(t) = w_0 \\ dp &= \theta\alpha\, ds + \sigma dz \text{ for } t < s < T, \quad p(t) = p_0 \end{aligned}$$

where dz is Brownian motion and θ, σ are fixed constants. The goal is to minimize, among (nonanticipating) controls $\alpha(s)$, the expected cost

$$E\left\{\int_t^T [p(s)\alpha(s) + \theta\alpha^2(s)]\, ds + [p(T)w(T) + \theta w^2(T)]\right\}.$$

The optimal expected cost is the value function $u(w_0, p_0, t)$.

(a) Show that the HJB equation for u is

$$u_t + H(u_w, u_p, p) + \frac{\sigma^2}{2}u_{pp} = 0$$

for $t < T$, with Hamiltonian

$$H(u_w, u_p, p) = -\frac{1}{4\theta}(p + \theta u_p - u_w)^2.$$

The final value is of course

$$u(w, p, T) = pw + \theta w^2.$$

(b) Look for a solution of the form $u(w, p, t) = pw + g(t)w^2$. Show that g solves

$$\dot{g} = \frac{1}{4\theta}(\theta - 2g)^2$$

for $t < T$, with $g(T) = \theta$. Notice that u does not depend on σ, i.e. setting $\sigma = 0$ gives the same value function.

(c) Solve for g. (Hint: start by rewriting the equation for g, "putting all the g's on the left and all the t's on the right".)

(d) Show by direct examination of your solution that the optimal $\alpha(s)$ is constant.

(Food for thought: what happens if one takes the running cost to be $\int_t^T p(s)\alpha(s)\, ds$ instead of $\int_t^T p(s)\alpha(s) + \theta\alpha^2(s)\, ds$?)

2) The Section 7 notes discuss work by Bertsimas, Kogan, and Lo involving least-square replication of a European option. The analysis there assumes all trades are *self-financing*, so the value of the portfolio at consecutive times is related by

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}).$$

Let's consider what happens if trades are permitted to be non-self-financing. This means we introduce an additional control g_j, the amount of cash added to (if $g_j > 0$) or removed from (if $g_j < 0$) the portfolio at time j, and the portfolio values now satisfy

$$V_j - V_{j-1} = \theta_{j-1}(P_j - P_{j-1}) + g_{j-1}.$$

It is natural to add a quadratic expression involving the g's to the objective: now we seek to minimize

$$E\left[(V_N - F(P_N))^2 + \alpha \sum_{j=0}^{N-1} g_j^2\right]$$

where α is a positive constant. The associated value function is

$$J_i(V, P) = \min_{\theta_i, g_i; \ldots; \theta_{N-1}, g_{N-1}} E_{V_i = V,\, P_i = P}\left[(V_N - F(P_N))^2 + \alpha \sum_{j=i}^{N-1} g_j^2\right].$$

The claim enunciated in the Section 7 notes remains true in this modified setting: J_i can be expressed as a quadratic polynomial

$$J_i(V_i, P_i) = \bar{a}_i(P_i)|V_i - \bar{b}_i(P_i)|^2 + \bar{c}_i(P_i)$$

where $\bar{a}_i, \bar{b}_i$, and $\bar{c}_i$ are suitably-defined functions which can be constructed inductively. Demonstrate this assertion in the special case $i = N-1$, and explain how $\bar{a}_{N-1}, \bar{b}_{N-1}, \bar{c}_{N-1}$ are related to the functions $a_{N-1}, b_{N-1}, c_{N-1}$ of the Section 7 notes.

3) Consider scaled Brownian motion with drift, $dy = \mu dt + \sigma dw$, starting at $y(0) = 0$. The solution is of course $y = \mu t + \sigma w(t)$, so its probability distribution at time t is Gaussian with mean μt and variance $\sigma^2 t$. Show that solution $\hat{p}(\xi, t)$ obtained by Fourier transform in the Section 8 notes is consistent with this result.

4) Consider scaled Brownian motion with drift and jumps: $dy = \mu dt + \sigma dw + J dN$, starting at $y(0) = 0$. Assume the jump occurences are Poisson with rate λ, and the jump magnitudes J are Gaussian with mean 0 and variance δ^2. Find the probability distribution of the process y at time t. (*Hint*: don't try to use the Fourier transform. Instead observe that you know, for any n, the probability that n jumps will occur before time t; and after conditioning on the number of jumps, the distribution of y is a Gaussian whose mean and variance are easy to determine. Assemble these ingredients to give the density of y as an infinite sum.)