PDE for Finance, Spring 2003 – Homework 5
Distributed 4/7/03, due 4/21/03.

Problem 1 is a classic example (due to Merton) of optimal asset allocation. Problems 2-4 reinforce our discussion of optimal stopping and American options. Problem 5 displays the power of dynamic programming for solving a different type of optimal stopping problem (one that's intrinsically discrete).

1) Consider the following asset-allocation problem. Two investment opportunities are available. One is risk-free, earning (constant) interest r. The other is lognormal, with (constant) drift μ and volatility σ, i.e. it satisfies $dp = \mu p ds + \sigma p dw$. You start at time t by investing wealth x. Your control is the weighting of your portofolio between these two assets, i.e.

$$\alpha(s) = \text{fraction of wealth invested in the risky asset at time } s$$

subject to $0 \leq \alpha \leq 1$. You never withdraw from or add to the portfolio, and you have a fixed horizon T. Your goal is to maximize the utility of your portfolio value at time T; in other words, your value function is

$$u(x,t) = \max_{\alpha(s)} E_{y(t)=x}\left[h(y(T))\right]$$

where $y(s)$ is the value of the portfolio at time s.

(a) Find the HJB equation satisfied by u.

(b) Find the solution – and the optimal investment strategy – if your utility is $h(y) = y^\gamma$ with $0 < \gamma < 1$.

(c) Find the solution – and the optimal investment strategy – if your utility is $h(y) = \log y$.

2) Example 2 of the Section 6 notes discusses when to sell a stock. The goal proposed in the notes was to maximize the discounted wealth realized by the sale, i.e.

$$\max_{\tau} E_{y(0)=x}\left[e^{-r\tau}(x-a)\right]$$

A different goal would be to maximize the discounted *utility* of wealth realized by the sale, i.e.

$$\max_{\tau} E_{y(0)=x}\left[e^{-r\tau}h(x-a)\right]$$

where h is your utility.

(a) Consider the utility $h(y) = y^\gamma$ with $0 < \gamma < 1$. (This is concave only for $y > 0$, but that's OK – it would clearly be foolish to sell at a price that realizes a loss.) Find the value function and the optimal strategy.

(b) The example in the notes corresponds to $\gamma = 1$. Using $\gamma < 1$ corresponds to introducing risk-averseness, and decreasing γ corresponds to increasing the risk-averseness. How is this reflected in the γ-dependence of the optimal strategy?

3) In Example 2 of the Section 6 notes we assumed $\mu < r$. Let's explore what happens when $\mu \geq r$. All other conventions of Example 2 remain in effect: the asset price satisfies $dy = \mu y dt + \sigma y dw$ and the value function is $u(x) = \max_\tau E_{y(0)=x}[e^{-r\tau}(y(\tau) - a)]$.

(a) Show that if $\mu > r$ then $u = \infty$.

(b) Show that if $\mu = r$ then $u(x) = x$.

(Hint: consider the value associated with sales threshold h, as $h \to \infty$.)

4) For a lognormal underlying with continuous dividend yield d, the risk-neutral process is $dy = (r - d)y dt + \sigma y dw$. The value of a perpetual American call with strike K is thus

$$u(x) = \max_\tau E_{y(0)=x}\left[e^{-r\tau}(y(\tau) - K)_+\right]$$

where r is the risk-free rate.

(a) Find the value of this option, and the optimal exercise rule, for $d > 0$.

(b) Show that as $d \to 0$ the value approaches $u(x) = x$.

5) [from Dimitri Bertsekas, *Dynamic Programming: Deterministic and Stochastic Models*, Chapter 2, problem 19]. A driver is looking for a parking place on the way to his destination. Each parking place is free with probability p, independent of whether other parking spaces are free or not. The driver cannot observe whether a parking place is free until he reaches it. If he parks k places from his destination, he incurs a cost k. If he reaches the destination without having parked the cost is C.

(a) Let F_k be the minimal expected cost if he is k parking places from his destination. Show that

$$\begin{aligned} F_0 &= C \\ F_k &= p\min[k, F_{k-1}] + qF_{k-1}, \quad k = 1, 2, \ldots \end{aligned}$$

where $q = 1 - p$.

(b) Show that an optimal policy is of the form: never park if $k \geq k^*$, but take the first free place if $k < k^*$, where k is the number of parking places from the destination, and k^* is the smallest integer i satisfying $q^{i-1} < (pC + q)^{-1}$.