PDE for Finance, Spring 2003 – Homework 1
Distributed 1/27/03, due 2/10/03.

1) Consider a diffusion $dy = f(y)ds + g(y)dw$, with initial condition $y(0) = x$. Suppose $u(x)$ solves the PDE

$$fu_x + \tfrac{1}{2}g^2 u_{xx} - q(x)u = 0 \quad \text{for } a < x < b, \text{ with } u = 1 \text{ at } x = a, b$$

for some function $q(x)$. Show that

$$u(x) = E_{y(0)=x}\left[e^{-\int_0^\tau q(y(s))ds}\right]$$

where τ is the exit time from $[a, b]$. (You should assume that $E[\tau] < \infty$.)

2) Consider the lognormal random walk

$$dy = \mu y dy + \sigma y dw$$

starting at $y(0) = x$. Assume $\mu \neq \frac{1}{2}\sigma^2$. The Section 1 notes examine the mean exit time from an interval $[a, b]$ where $0 < a < x < b$. There we used the PDE for the mean exit time

$$\mu x u_x + \tfrac{1}{2}\sigma^2 x^2 u_{xx} = -1 \quad \text{for } a < x < b \tag{1}$$

with boundary conditions $u(a) = u(b) = 0$ to derive an explicit formula for u.

(a) Show that the general solution of (1), without taking any boundary conditions into account, is

$$u = \frac{1}{\frac{1}{2}\sigma^2 - \mu}\log x + c_1 + c_2 x^\gamma$$

with $\gamma = 1 - 2\mu/\sigma^2$. Here c_1 and c_2 are arbitrary constants. [The formula given in the notes for the mean exit time is easy to deduce from this fact, by using the boundary conditions to solve for c_1 and c_2; however you need not do this calculation as part of your homework.]

(b) Argue as in the notes to show that the mean exit time from the interval (a, b) is finite. (Hint: mimic the argument used to answer Question 3, using $\phi(y) = \log y$.)

(c) Let p_a be the probability that the process exits at a, and $p_b = 1 - p_a$ the probability that it exits at b. Give an equation for p_a in terms of the barriers a, b and the initial value x. (Hint: mimic the argument used in the answer to Question 4, using $\phi(y) = y^\gamma$.) How does p_a behave in the limit $a \to 0$?

3) Consider a diffusion $dy = f(y)ds + g(y)dw$ starting at x at time 0, with $a < x < b$. Let τ be its exit time from the interval $[a, b]$, and assume $E[\tau] < \infty$.

(a) Let $u_a(x)$ be the probability it exits at a. Show that u_a solves $fu_x + \frac{1}{2}g_2 u_{xx} = 0$ with boundary conditions $u_a(a) = 1, u_a(b) = 0$.

(b) Apply this method to Problem 2(c). Is this approach fundamentally different from the one indicated by the hint above?

4) Consider once again a diffusion $dy = f(y)ds + g(y)dw$ starting at x at time 0. We know the mean arrival time to the boundary $v(x) = E[\tau]$ satisfies $fv_x + \frac{1}{2}g^2 v_{xx} = -1$ with $v = 0$ at $x = a, b$. Now consider the *second moment* of the arrival time $h(x) = E[\tau^2]$. Show that it satisfies $fh_x + \frac{1}{2}g^2 h_{xx} = -2v(x)$, with $h = 0$ at $x = a, b$.

5) Examine the analogues of Problem 2(a)–(c) when $\mu = \frac{1}{2}\sigma^2$. (Hint: notice that $xu_x + x^2 u_{xx} = u_{zz}$ with $z = \log x$.)

6) Let $w(t)$ be standard Brownian motion, starting at 0. Let τ_n be the first time w exits from the interval $[-n, 1]$, and let τ_∞ the the first time it reaches $w = 1$.

(a) Find the expected value of τ_n, and the probability that the path exits $[-n, 1]$ at $-n$.

(b) Verify by direct evaluation that $w(\tau_n)$ has mean value 0. (This must of course be true, since $E\left[\int_0^{\tau_n} dw = 0\right]$ by Dynkin's theorem.)

(c) Conclude from (a) that $E[\tau_\infty] = \infty$.

(d) Show that τ_∞ is almost-surely finite.