PDE for Finance Final Exam Questions
Spring 2003 – Professor Kohn

1) Give a probabilistic interpretation for the solution of each PDE. (You must justify your answer to receive full credit.)

(a) $u_t + f(x)u_x + \frac{1}{2}g^2(x)u_{xx} = 0$ for $t < T$ and $x \in R$, with final-time condition $u(x,T) = \phi(x)$.

(b) $f(x)u_x + \frac{1}{2}g^2(x)u_{xx} = -1$ on the interval $a < x < b$, with $u = 0$ at the boundary points $x = a, b$.

2) This problem concerns the explicit solution formulas for the linear heat equation in a half-space and a bounded interval.

(a) The solution of

$$u_t = u_{xx} \quad \text{for } t > 0 \text{ and } x > 1, \text{ with } u = (x-1)^3 \text{ at } t = 0 \text{ and } u = 0 \text{ at } x = 1$$

can be expressed as $u(x,t) = \frac{1}{\sqrt{4\pi t}} \int e^{-|x-y|^2/4t}\phi(y)\,dy$. What is $\phi(y)$?

(b) The solution of

$$u_t = u_{xx} \quad \text{for } t > 0 \text{ and } 0 < x < 1, \text{ with } u = g(x) \text{ at } t = 0 \text{ and } u = 0 \text{ at } x = 0, 1$$

can be expressed as $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$. Find $a_n(t)$ in terms of g.

3) This problem concerns the arrival time at the boundary, for a random walker solving $dy = f dt + g dw$ on the interval $[a,b]$.

(a) Let $G(x,y,t)$ be the probability, starting from x at time 0, of being at y at time t without having yet hit the boundary. What version of the forward Kolmogorov equation does G solve?

(b) Express, as an integral involving G_t, the "first passage time density to the boundary," i.e. the probability that the process, starting from $a < x < b$, first hits the boundary at time t.

(c) Using your answers to (a) and (b) and some further manipulation, show that

$$\text{first passage time density to the boundary} = -\tfrac{1}{2}\frac{\partial}{\partial y}(g^2 G(x,y,t))\Big|_{y=b} + \tfrac{1}{2}\frac{\partial}{\partial y}(g^2 G(x,y,t)\Big|_{y=a}.$$

4) Consider the following version of the Merton asset allocation problem:

- There is a risk-free asset, whose price satisfies $dp_1 = rp_1 ds$.
- There are two risky assets, whose prices p_2 and p_3 satisfy $dp_i = \mu_i p_i ds + \sigma_i p_i dw_i$ for $i = 2, 3$. We assume for simplicity that w_2 and w_3 are independent Brownian motions.
- Your controls are $\alpha_i(s) =$ the fraction of your wealth invested in asset i at time s, $i = 1, 2, 3$; note that $\alpha_1 + \alpha_2 + \alpha_3 = 1$.
- There is no consumption, and your goal is to optimize your expected utility of wealth at a predetermined time T. Your utility function is h.

Answer the following:

(a) What stochastic differential equation describes the evolution of your total wealth?

(b) Define an appropriate value function $u(x,t)$.

(c) Specify the Hamilton-Jacobi-Bellman equation and final-time condition u should satisfy.

(d) How does the value function determine the optimal asset allocations α_i?

5) In pricing a perpetual American put, we considered an underlying satisfying $dy = \mu y ds + \sigma y dw$ and the goal was to evaluate $\max_\tau E_{y(0)=x}\left[e^{-r\tau}(K - y(\tau))_+\right]$. Show that if v is a differentiable function with $v \geq (K-x)_+$ and $-rv + \mu x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} \leq 0$ for all x then v gives an upper bound:

$$E_{y(0)=x}\left[e^{-r\tau}(K - y(\tau))_+\right] \leq v(x)$$

for any bounded, nonanticipating stopping time τ.

6) This is a variant of the Bertsimas-Kogan-Lo least-squares-replication problem.considered in Section 7. It differs from the version in the notes in two ways: (i) the underlying has stochastic volatility; and (ii) the goal is not least-square replication but rather maximizing the utility of final-time wealth.

The underlying is a stock which can be traded at discrete times $i\Delta t$. Its price P_i and volatility σ_i at the ith time satisfy

$$\begin{aligned} \sigma_{i+1} &= \sigma_i + f(\sigma_i)\Delta t + g(\sigma_i)\phi_i\sqrt{\Delta t} \\ P_{i+1} &= P_i + \sigma_i P_i \psi_i \sqrt{\Delta t} \end{aligned}$$

where f and g are specified functions and ψ_i, ϕ_i are independent standard Gaussians (with mean 0 and variance 1).

You have sold an option on this stock with payoff $F(P_N)$, receiving cash V_0 in payment. Your goal is to invest this cash wisely, trading in a self-financing way, to maximize the expected utility of your final-time wealth $E[h(V_N - F(P_N))]$. Here h is your utility.

(a) Set this up as a discrete-time optimal control problem. What are the state variables? What is the control? Define an appropriate value function (call it J_i) at time $i\Delta t$. Be sure to specify the arguments of J_i, i.e. the variables it depends on.

(b) What is the value of J_N?

(c) Give a recursion relation that specifies J_i in terms of J_{i+1} for $i < N$.

7) Consider scaled Brownian motion with jumps: $dy = \sigma dw + J dN$, starting at $y(0) = x$. Assume the jump occurences are Poisson with rate λ, and the jumps have mean 0 and variance δ^2.

(a) Find $E[y^2(T)]$. (Hint: for a Poisson process with rate λ, the expected number of arrivals by time T is λT.)

(b) What backward Kolmogorov equation does part (a) solve?