PDE for Finance, Spring 2000 – Homework 5
Distributed 3/28/00, due 4/11/00. Solutions will be distributed 4/18/00.

The Black-Scholes-Merton theory of option pricing gives the value of an option as the (discounted) expected value of its payoff with respect to the risk-neutral dynamics. The following problems are intended to give insight about various types of options. However we don't attempt to identify the risk-neutral dynamics – that's Math Finance II material. Rather, we consider the expected value of the payoff (sometimes discounted, sometimes not) under some given dynamics. The books by Wilmott, Dewynne, and Howison are good sources for further reading; the hardback *Option Pricing* (on reserve in the CIMS library) contains more information than its paperback sibling.

1) Consider the lognormal process with drift μ and volatility σ (both assumed constant):

$$dy = \mu y ds + \sigma y dw.$$

(a) Show that the backward Kolmogorov equation solved by

$$u(x,t) = E_{y(t)=x}\left[\Phi(y(T))\right]$$

is

$$u_t + \mu x u_x + \tfrac{1}{2}\sigma^2 x^2 u_{xx} = 0$$

for $t < T$, with final-time condition $u(x,T) = \Phi(x)$.

(b) Verify using Ito's lemma that $y(s) = e^{z(s)}$ where z solves

$$dz = (\mu - \tfrac{1}{2}\sigma^2)ds + \sigma dw.$$

Deduce, using the backward Kolmogorov equation for z, that $v(x,t) = u(e^x, t)$ solves the constant-coefficient PDE

$$v_t + (\mu - \tfrac{1}{2}\sigma^2)v_x + \tfrac{1}{2}\sigma^2 v_{xx} = 0$$

for $t < T$, with final-time condition $v(x,T) = \Phi(e^x)$.

(c) Show that the substitution $v(x,t) = e^{-\alpha x - \beta t}w(x,t)$ leads to the linear heat equation $w_t + \tfrac{1}{2}\sigma^2 w_{xx} = 0$ when α and β are chosen so that $\alpha\sigma^2 = \mu - \tfrac{1}{2}\sigma^2$ and $\beta = \tfrac{1}{2}\sigma^2\alpha^2 - \alpha(\mu - \tfrac{1}{2}\sigma^2)$. (It's easy to solve for α and β, but I'm not asking you to do so.)

(d) Reconsider parts (a)-(b)-(c) when the definition of u is changed by introducing a discount term, with constant discount rate r:

$$u(x,t) = E_{y(t)=x}\left[e^{-r(T-t)}\Phi(y(T))\right].$$

Note: the role of the backward Kolmogorov equation is now played by the Feynman-Kac formula.

2) If the (instantaneous) interest rate satisfies a stochastic differential equation

$$dr = f(r, s)ds + g(r, s)dw$$

with $r(t) = x$, then the expected present value of one dollar received at time $T > t$ is

$$u(x, t) = E_{y(t)=x}\left[e^{-\int_t^T r(s)ds}\right].$$

What PDE does the Feynman-Kac formula give us for u? Be sure to specify the final-time condition for $u(x, T)$.

3) Consider the lognormal random walk with constant drift and volatility

$$dy = \mu y\, ds + \sigma y\, dw, \quad y(t) = x.$$

One type of barrier option has payoff $\Phi(y(T))$ (where Φ is a specified function) if $y(s)$ remains in a specified interval $a < y < b$, but payoff 0 if $y(s)$ touches either barrier a or b before time T. Of course we assume that the initial value x lies between a and b.

(a) Let $u(x, t)$ be the (undiscounted) expected payoff. What PDE does u solve? (Be sure to specify the final-time and boundary conditions. You need not actually solve the equation.)

(b) Let $v(x, t)$ be the probability that the random walk hits one of the barriers before time T. What PDE does v solve? (Be sure to specify the final-time and boundary conditions. You need not actually solve the equation.)

4) We continue to consider issues related to barrier options, for the lognormal random walk of problem 3. The Section 5 notes (page 6) consider the mean exit time u, i.e. the expected time at which y exits from (a, b). We used there the PDE

$$\mu x u_x + \tfrac{1}{2}\sigma^2 x^2 u_{xx} = -1 \quad \text{for } a < x < b \tag{1}$$

with boundary conditions $u(a) = u(b) = 0$ to derive an explicit formula for u. Assume for the following that $0 < a < b$, and (except for part d) that $\mu \neq \tfrac{1}{2}\sigma^2$.

(a) Show that the general solution of (1), without taking any boundary conditions into account, is

$$u = \frac{1}{\tfrac{1}{2}\sigma^2 - \mu}\log x + c_1 + c_2 x^\gamma$$

with $\gamma = 1 - 2\mu/\sigma^2$. Here c_1 and c_2 are arbitrary constants. [The formula given in the notes for the mean exit time is easy to deduce from this fact, by using the boundary conditions to solve for c_1 and c_2; however you need not do this calculation as part of your homework.]

(b) Argue as in the Section 5 notes to show that the mean exit time from the interval (a, b) is finite. (Hint: mimic the argument used in the answer to Question 3, using $\phi(y) = \log y$.)

(c) Let p_a be the probability that the process exits at a, and $p_b = 1 - p_a$ the probability that it exits at b. Give an equation for p_a in terms of the barriers a, b and the initial value x. (Hint: mimic the argument used in the answer to Question 4, using $\phi(y) = y^\gamma$.) How does p_a behave in the limit $a \to 0$?

(d) What happens when $\mu = \frac{1}{2}\sigma^2$?

5) An "Asian option" has the property that its payoff at maturity depends on the *average* value of the asset price from $0 < t < T$, as well as on the asset price itself. So it's of interest to evaluate

$$E_{y(0)=x}\left[\Phi\left(y(T), \frac{1}{T}\int_0^T y(s)ds\right)\right]$$

where Φ is a specified function of two (positive) variables. We assume for simplicity that y is described by the usual lognormal process

$$dy = \mu y ds + \sigma y dw.$$

(a) Show that if $z(s) = \int_t^s y(p)dp$ then $(y(s), z(s))$ solves

$$\begin{aligned} dy &= \mu y ds + \sigma y dw \\ dz &= y ds \end{aligned}$$

with initial condition $y(t) = x$, $z(t) = 0$.

(b) Show that

$$u(x, \xi; t) = E_{y(t)=x, z(t)=\xi}\left[\Phi\left(y(T), \frac{1}{T}z(T)\right)\right]$$

solves the backward Kolmogorov PDE

$$u_t + \mu x u_x + x u_\xi + \frac{1}{2}\sigma^2 x^2 u_{xx} = 0$$

for $t < T$, with final value $u(x, \xi; T) = \Phi(x, \xi/T)$ at $t = T$.

(c) Conclude that $E_{y(0)=x}\left[\Phi\left(y(T), \frac{1}{T}\int_0^T y(s)ds\right)\right]$ is equal to $u(x, 0; 0)$.

[Comment: the expected value of a function of $y(T)$ and $\max_{0<s<T} y(s)$ can be valued by a similar technique; this is related to the pricing of "lookback options." See Wilmott, Howison, and Dewynne.]