PDE for Finance, Spring 2000 – Homework 3. Distributed 2/22/00, due 3/7/00.

1) Our geometric Example 2 gave $|\nabla u| = 1$ in D (with $u = 0$ at ∂D) as the HJB equation associated with starting at a point x in some domain D, traveling with speed at most 1, and arriving at ∂D as quickly as possible. Let's consider what becomes of this problem when we introduce a little noise. The state equation becomes

$$dy = \alpha(s)ds + \epsilon dw, \quad y(0) = x,$$

where $\alpha(s)$ is a (non-anticipating) control satisfying $|\alpha(s)| \le 1$, y takes values in R^n, and each component of w is an independent Brownian motion. Let $\tau_{x,\alpha}$ denote the arrival time:

$$\tau_{x,\alpha} = \text{time when } y(s) \text{ first hits } \partial D,$$

which is of course random. The goal is now to minimize the *expected* arrival time at ∂D, so the value function is

$$u(x) = \min_{|\alpha(s)|\le 1} E_{y(0)=x}\left\{\tau_{x,\alpha}\right\}.$$

(a) Show, using an argument similar to that in the Section 3 notes, that u solves the PDE

$$1 - |\nabla u| + \tfrac{1}{2}\epsilon^2 \Delta u = 0 \quad \text{in } D$$

with boundary condition $u = 0$ at ∂D.

(b) Your answer to (a) should suggest a specific feedback strategy for determining $\alpha(s)$ in terms of $y(s)$. What is it?

2) Let's solve the differential equation from the last problem explicitly, for the special case when $D = [-1, 1]$:

$$\begin{aligned} 1 - |u_x| + \tfrac{1}{2}\epsilon^2 u_{xx} &= 0 \quad \text{for } -1 < x < 1 \\ u &= 0 \quad \text{at } x = \pm 1. \end{aligned}$$

(a) Assuming that the solution u is unique, show it satisfies $u(x) = u(-x)$. Conclude that $u_x = 0$ and $u_{xx} < 0$ at $x = 0$. Thus u has a maximum at $x = 0$.

(b) Notice that $v = u_x$ solves $1 - |v| + \delta v_x = 0$ with $\delta = \frac{1}{2}\epsilon^2$. Show that

$$\begin{aligned} v &= -1 + e^{-x/\delta} \qquad \text{for } 0 < x < 1 \\ v &= +1 - e^{x/\delta} \qquad \text{for } -1 < x < 0. \end{aligned}$$

Integrate once to find a formula for u.

(c) Verify that as $\epsilon \to 0$, this solution approaches $1 - |x|$.

[Comment: the assumption of uniqueness in part (a) is convenient, but it can be avoided. Outline of how to do this: observe that any critical point of u must be a local maximum (since $u_x = 0$ implies $u_{xx} < 0$). Therefore u has just one critical point, say x_0, which is a maximum. Get a formula for u by arguing as in (b). Then use the boundary condition to see that x_0 had to be 0.]

3) Let's consider what becomes of Merton's optimal investment and consumption problem if there are two risky assets: one whose price satisfies $dp_2 = \mu_2 p_2 dt + \sigma_2 p_2 dw_2$ and another whose price satisfies $dp_3 = \mu_3 p_3 dt + \sigma_3 p_3 dw_3$. To keep things simple let's suppose w_2 and w_3 are independent Brownian motions. It is natural to assume $\mu_2 > r$ and $\mu_3 > r$ where r is the risk-free rate. (Why?) Let $\alpha_2(s)$ and $\alpha_3(s)$ be the proportions of the investor's total wealth invested in the risky assets at time s, so that $1 - \alpha_2 - \alpha_3$ is the proportion of wealth invested risk-free. Then the investor's wealth satisfies

$$dy = (1 - \alpha_2 - \alpha_3) yr\,ds + \alpha_2 y(\mu_2 ds + \sigma_2 dw_2) + \alpha_3 y(\mu_3 ds + \sigma_3 dw_3).$$

(Be sure you understand this; but you need not explain it on your solution sheet.) Use the power-law utility: the value function is thus

$$u(x,t) = \max_{\alpha_2,\alpha_3,\beta} E_{y(t)=x}\left[\int_t^\tau e^{-\rho s}\beta^\gamma(s)\,ds\right]$$

where τ is the first time $y(s) = 0$ if this occurs, or $\tau = T$ otherwise.

(a) Derive the HJB equation.

(b) What is the optimal investment policy (the optimal choice of α_2 and α_3)? What restriction do you need on the parameters to be sure $\alpha_2 > 0$, $\alpha_3 > 0$, and $\alpha_2 + \alpha_3 < 1$?

(c) Find a formula for $u(x,t)$. [Hint: the nonlinear equation you have to solve is not really different from the one considered in Section 3.]

4) Problem 8 of Homework 2 was a special case of the deterministic "linear quadratic regulator" problem. Here is the analogous stochastic problem. The state is $y(s) \in R^n$, and the control is $\alpha(s) \in R^n$. There is no pointwise restriction on the possible value of $\alpha(s)$. The evolution law is

$$dy = (Ay + \alpha)ds + \epsilon dw,$$

where w is a vector-valued Brownian motion (each component is a scalar-valued Brownian motion, and different components are independent). The initial condition is $y(t) = x$, and the goal is to minimize (among nonanticipating controls) the expected cost

$$E_{y(t)=x}\left\{\int_t^T [|y(s)|^2 + |\alpha(s)|^2]\,ds + |y(T)|^2\right\}.$$

The interpretation is similar to the deterministic case: we prefer $y = 0$ for $t < s < T$ and at the final time T, but we also prefer not to use too much control. The new element is that the state keeps getting jostled by the noise ϵdw.

(a) Find the associated HJB equation. Explain why the relation $\alpha(s) = -\frac{1}{2}\nabla u(y(s)$ should hold for the optimal control. (Same relation as in the deterministic case!)

(b) Look for a solution of the form

$$u(x,t) = \langle K(t)x, x\rangle + q(t)$$

where $K(t)$ is symmetric-matrix-valued and $q(t)$ is scalar-valued. Show that this u solves the HJB equation exactly if

$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$

(same as the deterministic case), and

$$\frac{dq}{dt} = -\epsilon^2 \operatorname{tr} K(t) \text{ for } t < T, \quad q(T) = 0.$$

(c) Show that $K(t)$ is positive definite. (Hint: its quadratic form is the value function of the deterministic control problem.) Conclude that $q(t) > 0$ for $t < T$.

(d) Show by a verification argument that this u is indeed the value function of the control problem.

[Comment: in this setting the control law for the stochastic case, $\alpha(s) = -K(s)y(s)$, is the same as for the deterministic one. However the expected cost is higher due to the term $q(t)$.]