PDE for Finance, Spring 2000: Some types of questions that might be on the final exam. This list is exemplary, not comprehensive.

- Section 1 material:
  - Consider the following optimal control problem.... Identify an appropriate value function, and specify the Hamilton-Jacobi equation it solves. Assuming you know the value function, describe an optimal control strategy.
  - Consider the following optimal control problem ... with value function u, and the associated Hamilton-Jacobi equation .... Show that if v is differentiable and it solves the Hamilton-Jacobi equation ... then $u \leq v$.
- Section 2 material:
  - Consider the following optimal control problem .... What does the Pontryagin Maximum Principle tell us about its solution?
  - Consider the final-value problem $u_t + H(\nabla u) = 0$ for $t < T$ with $u = g$ at $t = T$. Assume H is convex. For what optimal control problem is u the value function? How does this lead to the Hopf-Lax solution formula for u?
- Section 3 material:
  - Consider the following stochastic optimal control problem... Identify an appropriate value function, and specify the Hamilton-Jacobi-Bellman equation it solves. What can you say about an optimal control policy?
  - Consider the Merton portfolio problem with utility c^γ (statement of problem here). Show (without finding or using the explicit solution) that the value function satisfies $u(ax,t) = a^\gamma u(x,t)$ for any a.
  - Explain the following statement: "Perturbing a deterministic optimal control problem by introducing a little noise in the dynamics corresponds to adding a small diffusion term to the HJB equation."
  - Consider the following stochastic control problem ... with value function u, and the associated Hamilton-Jacobi-Bellman equation ... Show that if v is differentiable and it solves the equation ... then $u \leq v$.
- Section 4 material:
  - Consider the following discrete-time stochastic dynamic programming problem.... Let $J_i(w_i)$ be the optimal expected value starting at time i, if the state at time i is w_i. What relation does the dynamic programming principle give you between J_i and J_{i-1}?
- Section 5 material:
  - Consider the stochastic differential equation $dy = f ds + g dw$ with infinitesimal generator $\mathcal{L}u = fu_x + (1/2)g^2 u_{xx}$. Show that if $\phi(x,t)$ satisfies $\phi_t + \mathcal{L}\phi \leq 0$ then $E_{y(t)=x}[\phi(y(T),T)] \leq \phi(x,t)$ for every $T > t$.
  - What PDE describes the probability that ....? (e.g. like HW 5 problem 3).
  - Consider the pde ..... For what stochastic process is it the forward Kolmogorov equation? What is the probabilistic interpretation of its solution? Is this PDE the backward Kolmogorov equation of any stochastic process?

  - Consider the stochastic differential equation.... Specify its forward and backward Kolmogorov equations. Show that if u solves the backward equation then $E[u(y(s), s)]$ is independent of s (you should use the fact that the evolving probability density satisfies the forward equation).

- Section 6 material:
  - The PDE $\rho_s - \frac{1}{2}\sigma^2\rho_{zz} = 0$ is the forward Kolmogorov equation for the stochastic PDE $dy = \sigma dw$. Use this fact – and basic properties of Brownian motion – to deduce the solution formula giving $\rho(z, s)$ in terms of the initial data $\rho_0(z) = \rho(z, 0)$.
  - Suppose $u_t - u_{xx} = 0$ for $x \in R$ and $t > 0$, with $u = u_0(x)$ at $t = 0$ and reasonable growth as $|x| \to \infty$. Suppose furthermore that u_0 is a convex function of x, i.e. $u_{0xx} \geq 0$. Show that $u_{xx} \geq 0$ for all x and all $t > 0$. Deduce that $u_t \geq 0$ for all x and all $t > 0$.
  - Suppose $u_t - u_{xx} + a(x, t)u = 0$ for $0 < x < 1$, with $u(0, t) = u(1, t) = 0$ for all $t > 0$. Suppose in addition $u > 0$ at $t = 0$. Show that $u > 0$ for all t.
  - Consider the simple finite-difference scheme .... for solving $u_t = u_{xx}$. Why is it important that $\Delta t < (1/2)(\Delta x)^2$?