PDE for Finance Notes – Section1 addendum

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Discussion of the example formulated in Section 1 involving a money-market account, a high-yield account, and transaction costs. These notes are slightly incomplete at the very end; comments, corrections, and improvements are welcome. The problem and its solution are from S. Shreve, H.M. Soner, and G. Xu, "Optimal investment and consumption with two bonds and transaction costs," *Math. Finance* Vol. 1, No. 3, 1991, 53-84. That paper considers a much more general class of utility functions; restricting our attention to utility γ^p permits a much simpler treatment.

Notation. In Section 1 we said the goal was to maximize $\int_0^\infty e^{-s}\gamma^p(s)ds$. It's equivalent to maximize $\frac{1}{p}\int_0^\infty e^{-s}\gamma^p(s)ds$. This trivial change of goal replaces u by u/p and changes the Hamilton-Jacobi equation accordingly. To match the notation of Shreve, Soner, and Xu I'll work with the latter utility. The following figure visualizes the region permitted by the solvency constraints.

Homogeneity. The value function satisfies

$$u(\lambda x, \lambda y) = \lambda^p u(x, y) \tag{1}$$

for any $\lambda > 0$. This is a consequence of (i) pth power utility, and (ii) proportional transaction costs. In fact, consider any starting portfolio (x, y) and controls $(\alpha(s), \beta(s), \gamma(s))$, and let $(X(s), Y(s))$ be the associated portfolio trajectory. Then $(\lambda X(s), \lambda Y(s))$ is the portfolio trajectory associated with starting portfolio $(\lambda x, \lambda y)$ and controls $(\lambda\alpha(s), \lambda\beta(s), \lambda\gamma(s))$, and its payoff is λ^p times that of γ. This shows that

$$u(\lambda x, \lambda y) \geq \lambda^p u(x, y).$$

Applying this with λ replaced by λ^{-1} gives

$$u(x, y) \geq \lambda^{-p} u(\lambda x, \lambda y).$$

These combine to give (1).

The $x > 0$ solvency boundary. It's easy to see that if the initial portfolio (x, y) lies on the $x > 0$ part of the insolvency boundary then the only admissible policy is to liquidate immediately, transfering funds from money-market to pay off the short high-yield position, effectively jumping to $(0,0)$. Thus $u = 0$ on this part of the solvency boundary. In fact: our hypothesis is $(1-\mu)x + y = 0$, and combining the state equations gives

$$\frac{d}{ds}[(1-\mu)X + Y] = r[(1-\mu)X + Y] + (R-r)Y + [(1-\mu)^2 - 1]\beta - \gamma.$$

The right hand side is nonpositive at $s = 0$, but it cannot be negative because this would make the trajectory leave the solvency region. This forces $\beta = \gamma = 0$, but more: it forces the liquidation to take place immediately, since otherwise the $(R-r)Y$ term would push the solution out of the solvency region. This reveals an imprecision in the problem formulation: if we only permit continuous controls (as our notation suggests), then we should not permit initial portfolios on the $x > 0$ part of the solvency boundary.

Restrictions on parameters. It's natural to place some restrictions on the parameters. Here's why. Suppose the initial portfolio is on the $y > 0$ part of the solvency boundary. Then one possible strategy is to set $\alpha = \beta = 0$, and choose $\gamma(s)$ in such as way as to keep $(X(s), Y(s))$ on the the solvency boundary. To find γ, observe that our strategy implies $X(s) + (1-\mu)Y(s) = 0$, $Y(s) = ye^{Rs}$, and $\dot{X} = rX - \gamma$. A little algebra gives

$$\gamma(s) = (R-r)(1-\mu)ye^{Rs}$$

and this gives the payoff

$$\frac{1}{p}\int_0^\infty (R-r)^p(1-\mu)^p y^p e^{(Rp-1)s}\,ds = \frac{y^p(1-\mu)^p(R-r)^p}{p(1-Rp)}$$

provided $pR < 1$. The integral would be infinite if $pR \geq 1$. Thus *to have an everywhere-finite value function we must require*

$$pR < 1. \tag{2}$$

That was just one possible strategy; there are others. A second natural way of staying on the $y > 0$ solvency boundary is to transfer funds continuously from the money market account into the high yield account (really: borrow money at the money-market rate and use it to invest at the high-yield rate). Suppose this is done using

$$\alpha(s) = \frac{1-pR}{1-\mu}c_0 Y(s)$$

for some $c_0 > 0$. Then a bit of calculation gives

$$Y(s) = ye^{c_1 s} \quad \text{with } c_1 = R + (1-pR)c_0,$$

and the relations $X(s) + (1-\mu)Y(s) = 0$, $\dot{X} = rX - \gamma - \alpha$ determine that

$$\gamma(s) = \left\{(1-\mu)(R-r) - \frac{2\mu - \mu^2}{1-\mu}(1-pR)c_0\right\} ye^{c_1 s}.$$

One can look for the condition that the payoff be finite, and if it is finite then one can optimize c_0. But instead, let's look for the condition that this policy be no better than the one considered earlier, i.e. the condition that the optimal c_0 be $c_0 = 0$. For any c_0 the payoff is

$$\frac{1}{p}y^p \frac{\left\{(1-\mu)(R-r) - \frac{2\mu-\mu^2}{1-\mu}(1-pR)c_0\right\}^p}{(1-Rp)(1-c_0p)}$$

provided $c_0p < 1$. The condition that the optimal c_0 be 0 is

$$(1-\mu)(R-r) < \frac{2\mu-\mu^2}{1-\mu}(1-pR), \tag{3}$$

which is stronger than (2). We shall henceforth assume the stronger restriction (3).

The Hamilton-Jacobi-Bellman equation. The HJB equation is easy to derive. We revert to the generic Section 1 notation: the state equation is $dy/ds = f(y,\alpha)$ with $y(0) = x$, and the value function is $u(x) = \max_\alpha \int_0^\infty e^{-s}h(y,\alpha)\,ds$. Arguing in our usual heuristic way:

$$u(x) \geq h(x,a)\Delta t + e^{-\Delta t}u(x + f(x,a)\Delta t).$$

Taylor expanding u, taking the limit $\Delta t \to 0$, and optimizing in a gives

$$0 = \max_a\{h + f \cdot \nabla u\} - u.$$

Now we change back to the notation of our example to implement this. It says $H(x,y,u_x,u_y) - u = 0$ with

$$H = \max_{\alpha,\beta,\gamma>0}\left\{\frac{1}{p}\gamma^p + (rx - \alpha + (1-\mu)\beta - \gamma)u_x + (Ry + (1-\mu)\alpha - \beta)u_y\right\}$$

Finiteness of H requires

$$-u_x + (1-\mu)u_y \leq 0 \quad \text{with strict negativity implying } \alpha = 0;$$

$$(1-\mu)u_x - u_y \leq 0 \quad \text{with strict negativity implying } \beta = 0; \text{ and}$$

$$u_x \geq 0.$$

The optimal γ satisfies $\gamma^{p-1} = u_x$, whence $\frac{1}{p}\gamma^p - \gamma u_x = \frac{1-p}{p}u_x^{p/(p-1)}$. (We are assuming $0 < p < 1$.) With this substitution, and assuming the finiteness conditions above,

$$H = \frac{1-p}{p}u_x^{p/(p-1)} + rxu_x + Ryu_y.$$

The verification theorem. We might hope to guess the optimal investment policy. But how will we ever prove the guess is right? Answer: use the usual verification argument.

Suppose $v(x,y) \geq 0$ is defined and differentiable on the solvent region of the (x,y) plane. Suppose furthermore

$$\begin{aligned} -v_x + (1-\mu)v_y &\leq 0 \\ (1-\mu)v_x - v_y &\leq 0 \\ -v + \frac{1-p}{p} v_x^{p/(p-1)} + rxv_x + Ryv_y &\leq 0 \end{aligned}$$

in this region. Then the value function u satisfies

$$u(x,y) \leq v(x,y).$$

Any proposal of an investment policy gives a lower bound on the value function u. The verification argument gives an upper bound. If we can make them agree, then we've found u (and also an optimal investment policy). The proof of the verification theorem is easy: we start with the calculation

$$\begin{aligned} \frac{d}{ds} e^{-s} v(X(s),Y(s)) &= e^{-s}(-v + v_x \dot{X} + v_y \dot{Y}) \\ &= e^{-s}\left(-v + v_x[rX - \alpha + (1-\mu)\beta - \gamma] + v_y[RY + (1-\mu)\alpha - \beta]\right) \\ &= e^{-s}\left(-v + \alpha[(1-\mu)v_y - v_x] + \beta[(1-\mu)v_x - v_y] + v_x rX + v_y RY - v_x\gamma\right) \\ &\leq e^{-s}\left(-v + v_x rX + v_y RY - v_x \gamma\right) \end{aligned}$$

using the positivity of α and β. Now remember from the calculation of H:

$$\frac{1}{p}\gamma^p - \gamma v_x \leq \frac{1-p}{p} v_x^{p/(p-1)}$$

which can be written as

$$-\gamma v_x \leq \frac{1-p}{p} v_x^{p/(p-1)} - \frac{1}{p}\gamma^p.$$

So

$$\begin{aligned} \frac{d}{ds} e^{-s} v(X(s),Y(s)) &= e^{-s}\left(-\frac{1}{p}\gamma^p(s) + \frac{1-p}{p} v_x^{p/(p-1)} + v_x rX + v_y RY - v\right) \\ &\leq e^{-s}\left(-\frac{1}{p}\gamma^p(s)\right). \end{aligned}$$

Integrating from $s=0$ to $s=S$, then using that $v(X(S),Y(S)) \geq 0$, then sending $S \to \infty$, we get

$$-v(X(0),Y(0)) \leq -\frac{1}{p}\int_0^\infty e^{-s}\gamma^p(s)\,ds,$$

i.e. $v(x,y) \geq u(x,y)$ as desired.

The solution. We claim that under the parameter restriction (3) the value function u and the optimal policy are as follows:

- The solvent region is divided into two parts by a ray $x = h_0 y$.
- To the right of this ray the optimal policy is to move immediately to this ray by transfering funds from money-market to high-yield. Consequently, to the right of this ray u is constant along each line $(1-\mu)x + y = \text{constant}$. This wedge might be called the "transfer from money-market region".
- To the left of this ray the optimal policy is to do no transactions, and to consume at rate $\gamma = u_x^{1/(p-1)}$. This wedge might be called the "no transaction region". The portfolio will approach and eventually hit (in finite time) the $y > 0$ part of the solvency boundary.
- Once the portfolio is on the solvency boundary the optimal policy is to stay on this boundary, doing no transactions. The associated consumption law and the value of u along this boundary were described above in the paragraph concerning restrictions on the parameters.

Using the homogeneity property, we can determine u more or less explicitly as follows. The homogeneity property implies that

$$u(x, y) = y^p \phi(x/y)$$

for some function $\phi(\xi)$. The last bullet above determines u on the $y > 0$ part of the solvency boundary, so it determines $\phi(\mu - 1)$:

$$\phi(\mu - 1) = \frac{(1-\mu)^p (R-r)^p}{p(1-Rp)}. \tag{4}$$

The second bullet says that in the no-transaction region u solves $-u + \frac{1-p}{p} u_x^{p/(p-1)} + r x u_x + R y u_y = 0$. Since

$$u_x = y^{p-1} \phi'(x/y) \quad u_y = y^{p-1} \left[p\phi(x/y) - (x/y)\phi'(x/y) \right]$$

we deduce a first-order differential equation for $\phi(\xi)$:

$$-\phi + \frac{1-p}{p} (\phi')^{p/(p-1)} + r\xi\phi'(\xi) + Rp\phi - R\xi\phi'(\xi) = 0. \tag{5}$$

Remember we need $(1-\mu)u_y - u_x \leq 0$; in terms of ϕ this restriction reads

$$(1-\mu)\left[p\phi(\xi) - \xi\phi'(\xi)\right] - \phi'(\xi) \leq 0. \tag{6}$$

The parameter restriction assures that this is true "initially", at $\xi = \mu - 1$. We may treat (4) and (5) as an initial-value problem for ϕ, but we must stop when (6) reaches 0. This determines a critical value of ξ, which we named named h_0 in the first bullet. For $\xi > h_0$ the value of ϕ is determined by extending u to the "transfer from money-market" region so it is constant on lines $(1-\mu)x + y =$ constant, i.e. so that $(1-\mu)u_y - u_x = 0$.

To show that this u is really the value function (and that the associated policy is optimal) we have only to check that it satisfies the hypotheses of the verification theorem. This takes a bit of work:

- To show that $u \geq 0$ we note that $\phi(\mu - 1) > 0$, and (6) gives $(1-\mu)p\phi(\xi) < (1+\xi(1-\mu))\phi'(\xi)$. We have $\xi \geq \mu - 1$, so $1 + \xi(1-\mu) > 0$, so $\phi' > 0$ for $\mu - 1 < \xi < h_0$. This shows $u \geq 0$ in the "no transaction" region. Its positivity in the "transfer from money-market" region follows immediately.

- To show that $-u + \frac{1-p}{p}u_x^{p/(p-1)} + rxu_x + Ryu_y \leq 0$ in the "transfer from money-market" region, one takes advantage of the fact that this holds at the line $x = h_0 y$, the fact that since u is constant along each line $(1-\mu)x + y =$ constant, so are u_x and u_y. (Details left to you.)

- We need not verify that $(1-\mu)u_y - u_x \leq 0$, since this is clear from the construction of ϕ.

- The final inequality $(1-\mu)u_x - u_y \leq 0$ is obvious in the "transfer from money market" region, since there we have $(1-\mu)u_y - u_x = 0$. It should hold in the "no transaction" region too, and I'm sure it does (from Shreve, Soner, and Xu), but I don't yet see the proof. In terms of ϕ the task is to show that

$$[(1-\mu) + \xi]\phi'(\xi) - p\phi(\xi) \leq 0 \quad \text{for } \mu - 1 \leq \xi \leq h_0. \tag{7}$$

  This inequality is readily checked at the endpoints $\xi = \mu - 1$ and $\xi = h_0$, where we already have explicit formulas or relations between ϕ and ϕ'. But why does it hold in between? (The corresponding step in Shreve, Soner, and Xu uses concavity of the utility function. Is (7) a convex function of ξ?)