PDE for Finance – Homework 3, distributed 2/24/99, due 3/24/99.

1) This problem refers to Section 3 of the notes, subsection "Optimal portfolio selection and consumption," which gives Merton's explicit solution to the portfolio selection and consumption problem when the utility of consumption is $h(c) = c^\gamma$.

(a) Examine the infinite-time-horizon limit of our solution ($T \to \infty$ with t and x held fixed). Show that it solves the stationary (time-independent) Hamilton-Jacobi equation. What is the optimal infinite-time-horizon investment policy?

(b) Find the analogous explicit (time-dependent) solution when the utility of consumption is $h(c) = \log c$. (Hint: this is essentially the $\gamma \to 0$ limit of the case $h(c) = c^\gamma$. Why?)

2) This problem develops a continuous-time analogue of the simple Bertsimas & Lo model of "Optimal control of execution costs" presented in Section 3 of the notes. The state is (w, p), where w is the number of shares yet to be purchased and p is the current price per share. The control $\alpha(s)$ is the rate at which shares are purchased. The state equation is:

$$\begin{aligned} dw &= -\alpha\, ds \text{ for } t < s < T, \quad w(t) = w_0 \\ dp &= \theta\alpha\, ds + \sigma dz \text{ for } t < s < T, \quad p(t) = p_0 \end{aligned}$$

where dz is Brownian motion and θ, σ are fixed constants. The goal is to minimize, among (nonanticipating) controls $\alpha(s)$, the expected cost

$$E\left\{\int_t^T [p(s)\alpha(s) + \theta\alpha^2(s)]\, ds + [p(T)w(T) + \theta w^2(T)]\right\}.$$

The optimal expected cost is the value function $u(w_0, p_0, t)$.

(a) Show that the HJB equation for u is

$$u_t + H(u_w, u_p, p) + \frac{\sigma^2}{2}u_{pp} = 0$$

for $t < T$, with Hamiltonian

$$H(u_w, u_p, p) = -\frac{1}{4\theta}(p + \theta u_p - u_w)^2.$$

The final value is of course

$$u(w, p, T) = pw + \theta w^2.$$

(b) Look for a solution of the form $u(w, p, t) = pw + g(t)w^2$. Show that g solves

$$\dot{g} = \frac{1}{4\theta}(\theta - 2g)^2$$

for $t < T$, with $g(T) = \theta$. Notice that u does not depend on σ, i.e. setting $\sigma = 0$ gives the same value function.

(c) Solve for g.

(d) Show by direct examination of your solution that the optimal $\alpha(s)$ is constant.

(e) Give another proof that the optimal $\alpha(s)$ is constant, by examining the deterministic version of this control problem ($\sigma = 0$) and arguing as we did for the Hopf-Lax formula.

(Remark: a better choice of objective would be $E\left\{\int_t^T [p(s)\alpha(s) + \theta'\alpha^2(s)]\,ds + [p(T)w(T) + \theta'' w^2(T)]\right\}$ for some constants θ', θ'', since the state equation gives θ units of dollars/(share)^2, whereas the units of θ' and θ'' are different. Food for thought: what happens if one takes the running cost to be $\int_t^T p(s)\alpha(s)\,ds$ instead of $\int_t^T p(s)\alpha(s) + \theta\alpha^2(s)\,ds$?)

3) Problem 6 of Homework 1 was a special case of the deterministic "linear quadratic regulator" problem. Here is the analogous stochastic problem. The state is $y(s) \in R^n$, and the control is $\alpha(s) \in R^n$. There is no pointwise restriction on the possible value of $\alpha(s)$. The evolution law is

$$dy = (Ay + \alpha)ds + \epsilon dw,$$

where w is a vector-valued Brownian motion (each component is a scalar-valued Brownian motion, and different components are independent). The initial condition is $y(t) = x$, and the goal is to minimize (among nonanticipating controls) the expected cost

$$E_{y(t)=x}\left\{\int_t^T [|y(s)|^2 + |\alpha(s)|^2]\,ds + |y(T)|^2\right\}.$$

The interpretation is similar to the deterministic case: we prefer $y = 0$ for $t < s < T$ and at the final time T, but we also prefer not to use too much control. The new element is that the state keeps getting jostled by the noise ϵdw.

(a) Find the associated HJB equation. Explain why the relation $\alpha(s) = -\frac{1}{2}\nabla u(y(s), s)$ should hold for the optimal control. (Same relation as in the deterministic case!)

(b) Look for a solution of the form

$$u(x, t) = \langle K(t)x, x\rangle + q(t)$$

where $K(t)$ is symmetric-matrix-valued and $q(t)$ is scalar-valued. Show that this u solves the HJB equation exactly if

$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$

(same as the deterministic case), and

$$\frac{dq}{dt} = -\epsilon^2 \mathrm{tr} K(t) \text{ for } t < T, \quad q(T) = 0.$$

(c) Show that $K(t)$ is positive definite. (Hint: its quadratic form is the value function of the deterministic control problem.) Conclude that $q(t) > 0$ for $t < T$.

(d) Show by a verification argument that this u is indeed the value function of the control problem. (Hint: argue as in the Section 3 addendum.)

(Remark: in this setting the control law for the stochastic case, $\alpha(s) = -K(s)y(s)$, is the same as for the deterministic one. However the expected cost is higher due to the term $q(t)$.)

4) Here is the discrete-time analogue of the preceding problem. The state at step k is $y_k \in R^n$, and the control at step k is $\alpha_k \in R^n$. There is no pointwise restriction on the possible values of α_k. The evolution law is

$$y_{k+1} = Ay_k + \alpha_k + e_k$$

where the e_k's are independent, identically distributed random variables with mean value 0 and finite variance. We emphasize that e_k is independent of y_k and α_k. The initial condition is $y_0 = x$, and the goal is to minimize the expected cost

$$E_{y_0=x}\left\{\sum_{k=0}^{N-1}[|y_k|^2 + |\alpha_k|^2] + |y_N|^2\right\}.$$

Let $J_k(x)$ be the minimum expected cost if the initial stage is k and the initial state is x. Observe that $J_N(x) = |x|^2$.

(a) Write the Bellman equation relating J_k to J_{k+1}.

(b) Look for a solution of the form $J_k(x) = \langle K_k x, x\rangle + q_k$, where K_k is a symmetric matrix and q_k is a scalar. Show that K_k satisfies the following recurrence relation:

$$K_k = A^T\left[K_{k+1} - K_{k+1}(K_{k+1}+I)^{-1}K_{k+1}\right]A + I$$

with $K_N = I$. How is (the optimal) α_k related to y_k? What is the value of q_k?

(Remark: For much more about the discrete-time LQR problem see section 2.1 of Bertsekas.)

5) [from Bertsekas: chapter 2, problem 10]. You want to sell a house. An offer comes at the beginning of each day. Successive offers are independent, and each offer is x_j with probability p_j, $j = 1, \ldots, n$. (This fixes the probability distribution of the offers.) An offer not immediately accepted is not lost, but may be accepted at any later date. A maintenance cost c is incurred for each day that the house remains unsold. You wish to maximize your (undiscounted) sale price minus your maintenance costs. Assume the house must be sold on or before day N. Characterize the optimal policy.

6) [from Bertsekas: chapter 2, problem 12]. A gambler plays a game in which he may at each time k stake any amount $u_k \geq 0$ that does not exceed his current fortune x_k (defined to be his initial capital plus his gain or minus his loss thus far). He wins his stake back and as much more with probability p, where $\frac{1}{2} < p < 1$, and he loses his stake with probability $(1-p)$. Look for the strategy that maximizes $E\{\log x_N\}$, where x_N is his fortune after

N plays. Show that the optimal choice is to stake, at each play, $2p-1$ times the current fortune (i.e. $u_k = (2p-1)x_k$).

7) [from Bertsekas: chapter 2, problem 19]. A driver is looking for a parking place on the way to his destination. Each parking place is free with probability p, independent of whether other parking spaces are free or not. The driver cannot observe whether a parking place is free until he reaches it. If he parks k places from his destination, he incurs a cost k. If he reaches the destination without having parked the cost is C.

(a) Let F_k be the minimal expected cost if he is k parking places from his destination. Show that

$$\begin{aligned} F_0 &= C \\ F_k &= p\min[k, F_{k-1}] + qF_{k-1}, \quad k = 1, 2, \ldots \end{aligned}$$

where $q = 1-p$.

(b) Show that an optimal policy is of the form: never park if $k \geq k^*$, but take the first free place if $k < k^*$, where k is the number of parking places from the destination, and k^* is the smallest integer i satisfying $q^{i-1} < (pC+q)^{-1}$.