PDE for Finance – Homework 1, distributed 1/20/99, due 2/3/99.

Problems 1–4 are relatively easy. Problems 5 and 6 are more difficult. Problem 7 is relatively easy except for the "food for thought" questions at the end.

1) Consider the Hopf-Lax solution formula for the dynamic programming solution to $u_t + \frac{1}{2}|\nabla u|^2 = 0$ with $u = g$ at $t = T$ (presented in the Section 1 notes). Suppose

$$g(x) = \begin{cases} 0 & \text{for } x \in E \\ -\infty & \text{for } x \notin E \end{cases}$$

for some set E. Evaluate the Hopf-Lax formula to find $u(x,t)$.

2) Consider the minimum-cost version of the finite-horizon problem, with dynamical law $dy/ds = f(y(s), \alpha(s))$ with $y(t) = x$ and objective $\min_\alpha \left\{ \int_t^T h(y(s), \alpha(s))\, ds + g(y(T)) \right\}$. The controls are restricted as usual by $\alpha(s) \in A$. Show (by a heuristic argument similar to those of the Section 1 notes) that the value function satisfies $u_t + H(x, \nabla u) = 0$ with

$$H(x,p) = \min_{a \in A} \left\{ f(x,a) \cdot p + h(x,a) \right\}.$$

3) Consider the minimum-cost, finite-horizon problem of Problem 2, with two different choices of the final-time cost $g^{(1)}$ and $g^{(2)}$. Let $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ be the associated value functions. Show that

$$|u^{(1)}(x,t) - u^{(2)}(x,t)| \le \max_{y \in R^n} |g^{(1)} - g^{(2)}|$$

for every $x \in R$ and every $t < T$. (Thus small changes in g lead only to small changes in u. The same thing is true, with the same proof, for the finite-horizon utility maximization problem.)

4) The Section 1 notes give a "minimum time" dynamic programming problem whose value function solves $|\nabla u| = 1$ for $x \notin \mathcal{T}$ with $u = 0$ at $\partial\mathcal{T}$.

(a) Find a related dynamic programming problem whose value function (if smooth) should solve $|\nabla u| = 1$ for $x \notin \mathcal{T}$ with $u = g$ at $\partial\mathcal{T}$, where g is a specified function on $\partial\mathcal{T}$.

(b) Consider the 2D case, with $\mathcal{T}$ a planar region with smooth boundary $\partial\mathcal{T}$. Describe the optimal controls and paths, if g is smooth and its derivative (with respect to arc-length) on $\mathcal{T}$ satisfies $|g'| < 1$.

(c) What changes if $|g'|$ is bigger than 1 on some part of $\partial\mathcal{T}$?

5) Consider the following physically natural minimum-time problem. A 1D particle with mass 1 has position x_1 and velocity x_2 at time 0. You can control it by applying a force of magnitude less then or equal to 1. Your goal is to bring it to rest at the origin as quickly as possible.

(a) Show we are considering a minimum-time problem with dynamics

$$dy_1/ds = y_2, \quad dy_2/ds = \alpha(s),$$

control $\alpha(s) \in A = \{|a| \leq 1\}$ and target set $\mathcal{T} = \{0, 0\}$.

(b) Find the associated Hamilton-Jacobi-Bellman equation.

(c) Show that when $a = 1$ the state moves along one of the parabolas $y_1 = \frac{1}{2}y_2^2 + c$. Similarly, if $a = -1$ the state moves along one of the parabolas $y_1 = -\frac{1}{2}y_2^2 + c$. From which starting points can the state move along one of these parabolas and arrive at $y_1 = y_2 = 0$?

(d) Show the following "feedback control" drives any initial state (x_1, x_2) to $(0, 0)$: take $\alpha(s)$ to be the following function of the state $(y_1(s), y_2(s))$:

$$\alpha = \begin{cases} -1 & \text{if } y_1 > -\frac{1}{2}y_2|y_2| \\ 1 & \text{if } y_1 > 0 \text{ and } y_1 = -\frac{1}{2}y_2|y_2| \\ 1 & \text{if } y_1 < -\frac{1}{2}y_2|y_2| \\ -1 & \text{if } y_1 < 0 \text{ and } y_1 = -\frac{1}{2}y_2|y_2| \end{cases}$$

(See the figure to visualize this.)

Show moreover that this control achieves value

$$u(x) = \begin{cases} x_2 + 2(x_1 + x_2^2/2)^{1/2} & \text{if } x_1 \geq -\frac{1}{2}x_2|x_2| \\ -x_2 + 2(-x_1 + x_2^2/2)^{1/2} & \text{if } x_1 \leq -\frac{1}{2}x_2|x_2| \end{cases}$$

(e) Show by a suitable "verification argument" that the control specified in (d) is optimal.

6) This problem is a special case of the "linear-quadratic regulator" widely used in engineering applications. The state is $y(s) \in R^n$, and the control is $\alpha(s) \in R^n$. There is no pointwise restriction on the values of $\alpha(s)$. The evolution law is

$$dy/ds = Ay + \alpha, \quad y(t) = x,$$

for some constant matrix A, and the goal is to minimize

$$\int_t^T |y(s)|^2 + |\alpha(s)|^2 \, ds + |y(T)|^2.$$

(In words: we prefer $y = 0$ along the trajectory and at the final time, but we also prefer not to use too much control.)

(a) What is the associated Hamilton-Jacobi-Bellman equation? Explain why we might expect the relation $\alpha(s) = -\frac{1}{2}\nabla u(y(s))$ to hold along optimal trajectories.

(b) Since the problem is quadratic, it's natural to guess that the value function $u(x,t)$ takes the form
$$u(x,t) = \langle K(t)x, x\rangle$$
for some symmetric $n \times n$ matrix-valued function $K(t)$. Show that this u solves the Hamilton-Jacobi-Bellman equation exactly if
$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$
where I is the $n \times n$ identity matrix. (Hint: two quadratic forms agree exactly if the associated symmetric matrices agree.)

(c) Show by a suitable verification argument that this u is indeed the value function of the control problem.

7) [An example of nonexistence of an optimal control.] Consider the following control problem: the state is $y(s) \in R$ with $y(t) = x$; the control is $\alpha(s) \in R$; the dynamics is $dy/dt = \alpha$; and the goal is
$$\text{minimize} \int_t^T y^2(s) + (\alpha^2(s) - 1)^2.$$
The value function $u(x,t)$ is the value of this minimum.

(a) Show that when $x = 0$ and $t < T$, the value is $u(0,t) = 0$.

(b) Show that when $x = 0$ and $t < T$ there is no optimal control $\alpha(s)$.

[The focus on $x = 0$ is only because this case is most transparent; nonexistence occurs for other (x,t) as well. Food for thought: What is the Hamilton-Jacobi-Bellman equation? Is there a modified goal leading to the same Hamiltonian and value function, but for which optimal controls exist?]