Consider a solution of the 1D wave equation
\[ u_{tt} - u_{xx} = 0 \]

with compactly supported initial data
\[ u = f \text{ and } u_t = g \text{ at } t = 0. \]

We know that the sum of the “kinetic energy” \( k(t) = \int u_t^2 \, dx \) and the “potential energy” \( p(t) = \int u_x^2 \, dx \) is independent of \( t \). (Note that both are finite, since \( u \) is compactly supported in space at every time.) Show that when \( t \) is sufficiently large these two “energies” are equal, in other words \( k(t) = p(t) \).

Let \( \alpha \) be constant, \( \alpha \neq -1 \). Consider the wave equation on \( x > 0, t > 0 \) with the following data:
\[ u_{tt} - u_{xx} = 0 \text{ for } x > 0, t > 0 \]
\[ u_t = \alpha u_x \text{ at } x = 0 \]
\[ u = f(x) \text{ at } t = 0 \]
\[ u_t = g(x) \text{ at } t = 0. \]

(a) Assume that \( f \) and \( g \) vanish near \( x = 0 \). Give a formula for \( u \). (Hint: start with \( u = F(r + t) + G(r - t) \); find \( F \) and \( G \).)

(b) What happens when \( \alpha = -1 \)?

(c) Returning to the case \( \alpha \neq -1 \), let’s drop the condition that \( f \) and \( g \) vanish in a neighborhood of \( x = 0 \). What conditions on \( f \) and \( g \) assure that the solution (obtained as in part (a)) is \( C^2 \)?

Consider solutions of \( u_{tt} = \Delta u \) in \( R^3 \times R_+ \) which are radially symmetric in \( x \). Show that the general solution is
\[ u = \frac{F(r + t) + G(r - t)}{r} \]

where \( r = |x| \). Show that for initial data of the special form
\[ u = 0, \quad u_t = g(r) \text{ at } t = 0 \]

(with \( g \) an even function of \( r \)) the solution is
\[ u = \frac{1}{2r} \int_{r-t}^{r+t} \rho g(\rho) \, d\rho. \quad (1) \]

Consider the case when
\[ g(r) = \begin{cases} 
1 & 0 < r < a \\
0 & r > a.
\end{cases} \]

Show there is a discontinuity at the origin at time \( t = a \). (Assume \( u \) is represented by (1), though it is not \( C^2 \) so the derivation of this formula is not strictly speaking applicable.)

Let’s look further at the radial solutions of the wave equation discussed in Problem 3.
(a) Let $G(r) = 1$ for $r \leq 1$ and $0$ for $r > 1$. Where is $u(r,t) = G(r-t)/r$ nonzero? (This solution can be viewed as an “outgoing wave.”)

(b) Discuss the character of $G(r-t)/r$ for a general function $G$.

(c) Let $F(r) = 1$ for $r > 100$ and $F = 0$ for $r < 100$. Where is $u(r,t) = F(r+t)/r$ nonzero? (This solution can be viewed as an “incoming wave.”)

(d) Discuss the character of $F(r+t)/r$ for a general function $F$.

(5) The pde $u_{tt} - u_{xx} + m^2 u = 0$ with $m \neq 0$ is known as the (one-dimensional) Klein-Gordon equation.

(a) For what choice of potential energy do we have that the kinetic + potential energy is constant in time?

(b) Show that, as for the wave equation, if at time 0 we have $u = u_t = 0$ in the interval $(-a,a)$, then $u$ vanishes in the triangle $\{ |x| \leq a-t \}$.

(6) Let $u_{tt} = \Delta u$ in $\mathbb{R}^n \times \mathbb{R}_+$. Show that

$$\sigma = \left[ -2 u_t \nabla u, (u_t^2 + |\nabla u|^2 f) \right]$$

is divergence-free (as a vector field in $\mathbb{R}^{n+1}$). Use this to show that $u(x,t)$ depends only on the initial data in the ball $\{ y : |y-x| \leq t \}$.

(7) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and suppose $u$ solves

$$u_{tt} - \Delta u = f(x) \text{ for } x \in \Omega, \ t > 0.$$ 

(a) Suppose the boundary condition is $u = 0$ at $\partial \Omega$. Estimate $e(t) = \frac{1}{2} \int_{\Omega} u_t^2 + |\nabla u|^2 \, dx$ in terms of its value at time 0, and the $L^2$ norm of $f$. Your answer should have the character of a well-posedness result; in other words, it should show that if $e(0)$ and $\|f\|_{L^2}$ are small enough then $e(t)$ is small. (Hint: start by considering $\frac{d}{dt} e(t)$.)

(b) Same question, when the boundary condition is $du/dn = 0$ at $\partial \Omega$.

(c) Can you do something similar when the boundary condition is $du/dn + au = 0$ with $a > 0$? (Hint: You’ll want to change the definition of $e(t)$ in this case. The good choice of $e(t)$ should be independent of time when $f = 0$.)