PDE I – Problem Set 8. Distributed Wed 11/5/2014, due in class 11/18/2014.

(1) Show that if u is a C^2 harmonic function defined on a region in R^n, then

$$v(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$$

is harmonic on the region where it is defined. (Hint: while this can be done by writing the Laplacian in polar coordinates, an alternative – in my view easier – argument uses the fact that u is harmonic in a domain Ω if and only if $\int_\Omega \langle \nabla u, \nabla \phi \rangle = 0$ for all ϕ such that $\phi = 0$ at $\partial\Omega$.) [Remark: v is known as the *Kelvin transform* of u. When we discussed the Green's function of the Laplacian for a half-space and a unit ball, we used the reflection of the fundamental solution around the boundary in the case of a half-space, and the Kelvin transform of the fundamental solution in the case of the unit ball.]

(2) Let's apply the Kelvin transform to the behavior of a harmonic function defined in the complement of a ball.

(a) Suppose u is a uniformly bounded harmonic function defined on $R^3 \setminus B_1 = \{x \in R^3 : |x| > 1$. Assume further that "$u \to 0$ uniformly at ∞" in other words that for any $\epsilon > 0$ there exists M such that $|x| > M$ implies $|u(x)| < \epsilon$. Show that the Kelvin transform of u has a removable singularity at 0.

(b) Using the conclusion of part (a), show there is a constant C (depending on u) such that $|u(x)| \leq C|x|^{-1}$ and $|\nabla u| \leq C|x|^{-2}$ for all sufficiently large $|x|$.

(c) What if u is defined in $R^3 \setminus B_a$ for some $a \neq 1$. (Hint: rescale the preceding results.)

(d) Are similar results true in R^n for $n > 3$? What about in R^2?

(3) Consider the quadrant $\{x > 0, y > 0\}$ in the $x - y$ plane. What is the Green's function of the Laplacian in this domain?

(4) We know a variational principle for solving the Neumann boundary value problem $\Delta u = f$ in Ω with $\partial u / \partial n = g$ at $\partial\Omega$ provided f and g are consistent. (For the record: it is to minimize $\int_\Omega \frac{1}{2}|\nabla u|^2 + fu\,dx - \int_{\partial\Omega} gu\,dA$ over all $u : \Omega \to R$; recall that the boundary condition $\partial u/\partial n = g$ is a consequence of the first variation being zero at u.) This problem shows that one cannot solve that PDE problem by instead imposing the boundary condition as a constraint. For simplicity let's work in 1D, taking $\Omega = (0, 1)$; and let's take $f = 0$. *Here's the question*: show that for an $a, b \in R$, the (misguided) variational problem

$$\min_{u_x(0)=a,\, u_x(1)=b} \int_0^1 u_x^2$$

has minimum value 0. [Food for thought: why is it OK to fix $u|_{\partial\Omega}$, as we do for a Dirichlet boundary condition, though this problem shows that it is not OK to fix $\partial u/\partial n|_{\partial\Omega}$?]

(5) Use the convexity of

$$E[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}u^4\,dx$$

to prove that there can be at most one solution of $-\Delta u + u^3 = 0$ in Ω with a given Dirichlet boundary condition $u = g$ at $\partial\Omega$.

(6) Here is another example of a variational principle that leads to a PDE. Let Ω be a bounded domain in R^n, and consider the variational problem

$$\min_{\int_\Omega u\,dx=0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

Assume that a minimizer exists and is C^2. Show that it must be a Neumann eigenfunction of the Laplacian, i.e. a nonzero solution of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \text{ with } \partial u/\partial n = 0 \text{ at } \partial\Omega.$$

Conclude that the minimum value of this variational problem is equal to the smallest nonzero Neumann eigenvalue.

(7) A question about the finite element method:

(a) Explain why if u and v are piecewise linear on $[0,1]$, determined by their nodal values u_j, v_j at $x_j = j/N$, then integration gives

$$\int_0^1 uv\,dx = \frac{1}{N}\langle K\vec{u}, \vec{v}\rangle$$

where K is a symmetric matrix, $\vec{u} = (u_0, u_1, \ldots, u_N)$ and $\vec{v} = (v_0, v_1, \ldots, v_N)$. What is K?

(b) With the same notation as in (a), express $\int_0^1 u_x^2\,dx$ in terms of the nodal values of u.