
(1) Show that if \( u \) is a \( C^2 \) harmonic function defined on a region in \( \mathbb{R}^n \), then
\[
v(x) = |x|^{2-n}u \left( \frac{x}{|x|^2} \right)
\]
is harmonic on the region where it is defined. (Hint: while this can be done by writing the Laplacian in polar coordinates, an alternative – in my view easier – argument uses the fact that \( u \) is harmonic in a domain \( \Omega \) if and only if
\[
\int_{\Omega} \langle \nabla u, \nabla \phi \rangle = 0 \text{ for all } \phi \text{ such that } \phi = 0 \text{ at } \partial \Omega.
\]
[Remark: \( v \) is known as the Kelvin transform of \( u \). When we discussed the Green’s function of the Laplacian for a half-space and a unit ball, we used the reflection of the fundamental solution around the boundary in the case of a half-space, and the Kelvin transform of the fundamental solution in the case of the unit ball.]

(2) Let’s apply the Kelvin transform to the behavior of a harmonic function defined in the complement of a ball.

(a) Suppose \( u \) is a uniformly bounded harmonic function defined on \( \mathbb{R}^3 \setminus B_1 = \{ x \in \mathbb{R}^3 : |x| > 1 \} \). Assume further that "\( u \to 0 \) uniformly at \( \infty \)" in other words that for any \( \epsilon > 0 \) there exists \( M \) such that \( |x| > M \) implies \( |u(x)| < \epsilon \). Show that the Kelvin transform of \( u \) has a removable singularity at 0.

(b) Using the conclusion of part (a), show there is a constant \( C \) (depending on \( u \)) such that
\[
|u(x)| \leq C|x|^{-1} \text{ and } |\nabla u| \leq C|x|^{-2}
\]
for all sufficiently large \( |x| \).

(c) What if \( u \) is defined in \( \mathbb{R}^3 \setminus B_a \) for some \( a \neq 1 \). (Hint: rescale the preceding results.)

(d) Are similar results true in \( \mathbb{R}^n \) for \( n > 3 \)? What about in \( \mathbb{R}^2 \)?

(3) Consider the quadrant \( \{ x > 0, y > 0 \} \) in the \( x-y \) plane. What is the Green’s function of the Laplacian in this domain?

(4) We know a variational principle for solving the Neumann boundary value problem \( \Delta u = f \) in \( \Omega \) with \( \partial u/\partial n = g \) at \( \partial \Omega \) provided \( f \) and \( g \) are consistent. (For the record: it is to minimize \( \int_{\Omega} \frac{1}{2} |\nabla u|^2 + f u \, dx - \int_{\partial \Omega} gu \, dA \) over all \( u : \Omega \to \mathbb{R} \); recall that the boundary condition \( \partial u/\partial n = g \) is a consequence of the first variation being zero at \( u \).) This problem shows that one cannot solve that PDE problem by instead imposing the boundary condition as a constraint. For simplicity let’s work in 1D, taking \( \Omega = (0,1) \); and let’s take \( f = 0 \). Here’s the question: show that for an \( a, b \in \mathbb{R} \), the (misguided) variational problem
\[
\min_{u_x(0)=a, u_x(1)=b} \int_0^1 u_x^2 \, dx
\]
has minimum value 0. [Food for thought: why is it OK to fix \( u_{\mid \partial \Omega} \), as we do for a Dirichlet boundary condition, though this problem shows that it is not OK to fix \( \partial u/\partial n_{\mid \partial \Omega} \)?]

(5) Use the convexity of
\[
E[u] = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} u^4 \, dx
\]
to prove that there can be at most one solution of \( -\Delta u + u^3 = 0 \) in \( \Omega \) with a given Dirichlet boundary condition \( u = g \) at \( \partial \Omega \).
Here is another example of a variational principle that leads to a PDE. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), and consider the variational problem

\[
\min_{\int_{\Omega} u \, dx = 0} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} u^2 \, dx}.
\]

Assume that a minimizer exists and is \( C^2 \). Show that it must be a Neumann eigenfunction of the Laplacian, i.e. a nonzero solution of

\[-\Delta u = \lambda u \quad \text{in } \Omega, \quad \text{with } \partial u / \partial n = 0 \quad \text{at } \partial \Omega.
\]

Conclude that the minimum value of this variational problem is equal to the smallest nonzero Neumann eigenvalue.

A question about the finite element method:

(a) Explain why if \( u \) and \( v \) are piecewise linear on \([0, 1]\), determined by their nodal values \( u_j, v_j \) at \( x_j = j/N \), then integration gives

\[
\int_0^1 uv \, dx = \frac{1}{N} \langle K\vec{u}, \vec{v} \rangle
\]

where \( K \) is a symmetric matrix, \( \vec{u} = (u_0, u_1, \ldots, u_N) \) and \( \vec{v} = (v_0, v_1, \ldots, v_N) \). What is \( K \)?

(b) With the same notation as in (a), express \( \int_0^1 u_x^2 \, dx \) in terms of the nodal values of \( u \).