PDE I – Problem Set 7. Distributed Wed 10/29/2014, due in class 11/11/2014.

(1) If u is harmonic on $B_r(0) \subset R^n$ with $u = g$ at $|x| = r$, it can be represented using Poisson's formula:

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n}.$$

(For $n = 2$ we discussed this in class on 10/28. In all dimensions, it follows from the formula for the Green's function of a ball, which we'll discuss on 11/4.) Use this to show that if u is harmonic and nonnegative on $B_r(0)$ then

$$r^{n-2}\frac{r - |x|}{(r+|x|)^{n-1}} u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}} u(0). \tag{1}$$

[Remark: Harnack's inequality says that if u is a nonnegative harmonic function on a domain Ω, and ω is a subdomain whose closure does not meet $\partial\Omega$, then the sup and inf of u on ω are comparable, in the sense that $\sup_\omega u \le C \inf_\omega u$ for some constant C (depending on ω and Ω, but independent of u). For a proof based on the mean value principle see Section 2.2 of Evans. The estimate (1) leads easily to the special case of Harnack's inequality when ω and Ω are concentric spheres, with a constant C that's explicit in terms of their radii.]

(2) Recall that a periodic function on R^n (with period 1 in each variable) has a Fourier series:

$$u(x) = \sum_{k \in Z^n} \hat{u}(k) e^{2\pi i k \cdot x} .$$

Let's use this to study the inhomogeneous Laplace equation

$$\Delta u = f$$

with periodic boundary conditions (we assume f is periodic, and we seek a solution with u periodic):

(a) What consistency condition should f satisfy? Show by an energy argument that u is unique up to an additive constant.

(b) Express the Fourier series of u in terms of that of f.

(c) Show that for each i, j,

$$\int_Q \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \le C \int_Q |f|^2$$

where $Q = [0,1]^n$ is the period cell. Can you identify the optimal value of C?

(3) Suppose u is harmonic in the punctured ball B_r, and

$$\frac{|u(x)|}{|\Phi(x)|} \to 0 \text{ as } |x| \to 0$$

where Φ is the fundamental solution of the Laplacian. Show that the singularity is removable, i.e. u is actually harmonic in the entire ball. (Hint: Let w be the harmonic function in B_r with $w|_{\partial B_r} = u|_{\partial B_r}$; your task is to prove that $w(z) = u(z)$ for any $z \neq 0$. Consider $u - w \pm \varepsilon(\Phi - c)$, where $c = \Phi(r)$. Apply the maximum principle on a well-chosen annulus $\{\delta < |x| < r\}$.)

(4) Let Ω be a bounded domain in R^n, and consider the operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}$$

where $a_{ij}(x)$ and $b_i(x)$ are continuous and $a_{ij} = a_{ji}$. Assume moreover that there is a positive lower bound on the eigenvalues of a_{ij}, i.e. that $\sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq c_0|\xi|^2$ for all $x \in \Omega$ and all $\xi \in R^n$, for some $c_0 > 0$. Show that

(a) if u is C^2 and $Lu \geq 0$ in Ω then $\max_{x\in\Omega} u(x) = \max_{x\in\partial\Omega} u(x)$;

(b) if If u is C^2 and $Lu \leq 0$ in Ω then $\min_{x\in\Omega} u(x) = \min_{x\in\partial\Omega} u(x)$.

(Hint: consider, for sufficiently large λ, the function $u_\epsilon = u(x) \pm \epsilon e^{\lambda x_1}$.)

(5) Use problem 4 to show that if Ω is a bounded domain in R^n and $F : R^n \to R$ is smooth then there can be at most one solution of $\Delta u = F(\nabla u)$ with a given Dirichlet boundary condition $u = g$ at $\partial\Omega$. (Hint: By Taylor's theorem with remainder, $F(\xi) - F(\eta) = \left(\int_0^1 \nabla F(\eta + t(\xi - \eta))\, dt\right) \cdot (\xi - \eta)$.)