
(1) If $u$ is harmonic on $B_r(0) \subset \mathbb{R}^n$ with $u = g$ at $|x| = r$, it can be represented using Poisson’s formula:

$$u(x) = \frac{r^2 - |x|^2}{n \alpha(n) r} \int_{\partial B_r(0)} g(y) \frac{|x - y|^n}{|x - y|^n}.$$ 

(For $n = 2$ we discussed this in class on 10/28. In all dimensions, it follows from the formula for the Green’s function of a ball, which we’ll discuss on 11/4.) Use this to show that if $u$ is harmonic and nonnegative on $B_r(0)$ then

$$r^{-2} u(0) \leq u(x) \leq r^{-2} u(0).$$ 

[Remark: Harnack’s inequality says that if $u$ is a nonnegative harmonic function on a domain $\Omega$, and $\omega$ is a subdomain whose closure does not meet $\partial \Omega$, then the sup and inf of $u$ on $\omega$ are comparable, in the sense that $\sup_{\omega} u \leq C \inf_{\omega} u$ for some constant $C$ (depending on $\omega$ and $\Omega$, but independent of $u$). For a proof based on the mean value principle see Section 2.2 of Evans. The estimate (1) leads easily to the special case of Harnack’s inequality when $\omega$ and $\Omega$ are concentric spheres, with a constant $C$ that’s explicit in terms of their radii.]

(2) Recall that a periodic function on $\mathbb{R}^n$ (with period 1 in each variable) has a Fourier series:

$$u(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}(k) e^{2\pi i k \cdot x}.$$ 

Let’s use this to study the inhomogeneous Laplace equation

$$\Delta u = f$$

with periodic boundary conditions (we assume $f$ is periodic, and we seek a solution with $u$ periodic):

(a) What consistency condition should $f$ satisfy? Show by an energy argument that $u$ is unique up to an additive constant.

(b) Express the Fourier series of $u$ in terms of that of $f$.

(c) Show that for each $i, j$,

$$\int_Q \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \leq C \int_Q |f|^2$$

where $Q = [0, 1]^n$ is the period cell. Can you identify the optimal value of $C$?

(3) Suppose $u$ is harmonic in the punctured ball $B_r$, and

$$\frac{|u(x)|}{\Phi(x)} \to 0 \text{ as } |x| \to 0$$

where $\Phi$ is the fundamental solution of the Laplacian. Show that the singularity is removable, i.e. $u$ is actually harmonic in the entire ball. (Hint: Let $w$ be the harmonic function in $B_r$ with $w|_{\partial B_r} = u|_{\partial B_r}$; your task is to prove that $w(z) = u(z)$ for any $z \neq 0$. Consider $u - w \pm \varepsilon(\Phi - c)$, where $c = \Phi(r)$. Apply the maximum principle on a well-chosen annulus $\{ \delta < |x| < r \}$.)
(4) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and consider the operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}$$

where $a_{ij}(x)$ and $b_i(x)$ are continuous and $a_{ij} = a_{ji}$. Assume moreover that there is a positive lower bound on the eigenvalues of $a_{ij}$, i.e., that $\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, for some $c_0 > 0$. Show that

(a) if $u$ is $C^2$ and $Lu \geq 0$ in $\Omega$ then $\max_{x \in \Omega} u(x) = \max_{x \in \partial \Omega} u(x)$;

(b) if $u$ is $C^2$ and $Lu \leq 0$ in $\Omega$ then $\min_{x \in \Omega} u(x) = \min_{x \in \partial \Omega} u(x)$.

(Hint: consider, for sufficiently large $\lambda$, the function $u_{\epsilon} = u(x) \pm \epsilon e^{\lambda x_1}$.)

(5) Use problem 4 to show that if $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$ is smooth then there can be at most one solution of $\Delta u = F(\nabla u)$ with a given Dirichlet boundary condition $u = g$ at $\partial \Omega$. (Hint: By Taylor’s theorem with remainder, $F(\xi) - F(\eta) = \left(\int_0^1 \nabla F(\eta + t(\xi - \eta)) \, dt\right) \cdot (\xi - \eta)$.)