PDE I – Problem Set 6. Distributed Thurs 10/9/2014, due in class 10/28. *Problem 4b corrected 10/26 – a constant was missing before.*

Please note: You may bring one sheet of notes to the midterm (Tues 10/21). Its scope consists of: (a) everything in Lectures 1-5 and Problem Sets 1-5, plus (b) the parts of the Lecture 6 notes that we covered on 10/7, namely: the fundamental solution of Laplace's equation (pages 1-7 of those notes), finite difference approximation for Laplace's equation (page 13 of those notes), and the weak form of the maximum principle for Laplace's equation (page 14 of those notes).

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(1) (From Evans, Section 2.5.) Adjust the proof of the mean value theorem to show that for $n \geq 3$, if $-\Delta u = f$ on $B(0,r)$ and $u = g$ at the boundary, then

$$u(0) = \frac{1}{|\partial B(0,r)|} \int_{\partial B(0,r)} g \, \text{darea} + \frac{1}{n(n-2)\alpha_n} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f \, \text{dvol}.$$

(2) Show that the function $\Phi(x) = -\frac{1}{2}|x|$ is a fundamental solution of the 1D Laplacian, in the sense that for any compactly supported function $f : R \to R$, the function $u(x) = \int \Phi(x - y) f(y) \, dy$ solves $-u_{xx} = f$.

(3) One version of the "weak form" of the maximum principle is that:

- if $\Delta u \geq 0$ in Ω then u achieves its max at $\partial\Omega$, and
- if $\Delta u \leq 0$ in Ω then u achieves its min at $\partial\Omega$,

where Ω is a bounded domain in R^n. (You should know how to prove these, but I'm not asking you to write it up.) Using this, find an explicit constant C such that

$$\max_B |u| \leq C \max_B |f|$$

when $B = B_1(0)$ is the unit ball in R^n, and u solves the boundary value problem

$$\Delta u = f \text{ in } B \text{ with } u = 0 \text{ at } \partial B. \tag{1}$$

(4) Problem 3 concerns how the maximum principle can be used to prove well-posedness of the boundary value problem described by eqn (1). This problem concerns how the energy method gives an alternative approach to well-posedness.

(a) Poincare's inequality says that if $u = 0$ at $\partial\Omega$ then $\int_\Omega u^2 \leq C \int_\Omega |\nabla u|^2$ (here Ω is a bounded domain in R^n, and the constant C depends on Ω). Prove it. [Hint: we can extend u by 0 outside Ω, so it's defined in a cube in R^n. Therefore it suffices to prove the inequality when Ω is a cube.]

(b) Using Poincare's inequality, show that if u solves eqn (1) then $\int_\Omega |\nabla u|^2 \leq C \int_\Omega f^2$. [Hint: multiply the equation by u and integrate.]

(5) We proved the (weak form of the) maximum principle for harmonic functions on a bounded domain. When the domain is unbounded an additional hypothesis is needed; for example, in the halfspace $x_n > 0$ the linear function $u(x) = x_n$ is harmonic but doesn't achieve its maximum on the boundary. Let's focus for simplicity on the 2D halfspace $\Omega = \{(x_1, x_2) : x_2 > 0\}$. Show that if u is C^2 and harmonic on this Ω and continuous up to the boundary, and if in addition u is uniformly bounded from above, then $\max_\Omega u = \max_{\partial\Omega} u$. [Hint: for $\epsilon > 0$, consider the harmonic function $u(x) - \epsilon \log(x_1^2 + (x_2+1)^2)^{1/2}$. Apply the maximum principle to the region where $x_1^2 + (x_2+1)^2 < a^2$ and $x_2 > 0$, with a sufficiently large. Then let $\epsilon \to 0$.]