PDE I – Problem Set 5. Distributed Wed 10/1/2014, due by 5pm on Friday 10/17/2014. **No extensions!**

Please note: Our *midterm exam* is Tues 10/21, in the normal class location and time. You may bring one sheet of notes (8.5 × 11, both sides, any font). A solution sheet to Problem Set 5 will be distributed Friday evening 10/17; for this reason, there will be no extensions beyond 5pm on 10/17. Since there is no class on 10/14, you should turn in your paper by giving it to Professor Kohn, putting it in his WWH lobby mailbox, or by sending it to kohn@cims.nyu.edu. (For scans: please send a single file, not a photo of each page.)

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In this problem set, questions 1 and 2 reinforce our discussion of variational principles. Problems 3-7 explore connections between harmonic functions, Fourier series, and complex variables. Problems 8 and 9 are applications of the mean value principle.

(1) We showed in class (and in the Lecture 5 notes) that if Ω is a bounded domain in R^n and a (smooth enough) function u_* achieves

$$\min_{u=\phi \text{ at } \partial\Omega} \int_\Omega |\nabla u|^2 \, dx$$

then $\Delta u_* = 0$ in Ω. Show that the converse is also true: suppose that u_* is a solution of $\Delta u_* = 0$ in Ω with $u_* = \phi$ at $\partial\Omega$. Show that for any (smooth enough) function v with $v = \phi$ at $\partial\Omega$ we have $\int_\Omega |\nabla v|^2 \, dx \geq \int_\Omega |\nabla u_*|^2 \, dx$, with equality only when $v = u_*$.

(2) Show that if a (smooth enough) function u_* minimizes

$$E[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + \int_\Omega uf + \frac{1}{2}\beta \int_{\partial\Omega} |u|^2$$

then it solves $\Delta u_* = f$ in Ω, with the boundary condition $\frac{\partial u_*}{\partial n} + \beta u_* = 0$ at $\partial\Omega$.

(3) Let $f(\theta)$ be a periodic function on the unit circle in the plane, with Fourier series

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta).$$

Assume that $f_{\theta\theta}$ is uniformly bounded. Show that

$$u = \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)] r^n$$

solves Laplace's equation in the disk $r < 1$ with boundary condition f.

(4) Give a similar method for solving the Neumann problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in the disk } x^2 + y^2 < 1 \\ \partial u/\partial n &= g \quad \text{at the circle } x^2 + y^2 = 1. \end{aligned}$$

(5) Now consider solving $\Delta u = 0$ with both Dirichlet and Neumann data imposed at the unit circle:

$$u = f \text{ and } \frac{\partial u}{\partial n} = g \text{ at } x^2 + y^2 = 1.$$

Let's explore whether there a solution in the *punctured* disk $0 < x^2 + y^2 < 1$.

  (a) Show that there are plenty of examples where the answer is yes. (Hint: consider a complex function that's analytic in the punctured disk).

  (b) Show that there are plenty of examples where the answer is no (here, too, I suggest using your knowledge of complex variables).

  (c) How should the Fourier series of f and g be related, if there is to be a solution that's harmonic on the whole disk $x^2 + y^2 < 1$?

(6) For any α between 0 and 2π, consider Laplace's equation $\Delta u = 0$ in the pie-shaped region $\Omega = \{re^{i\theta} : 0 < r < 1, 0 < \theta < \alpha\}$, with $\frac{\partial u}{\partial n} = 0$ at the straight parts of the boundary (the segments $\theta = 0$ and $\theta = \alpha$) and $u = f(\theta)$ at the curved part (the arc $e^{i\theta}, 0 < \theta < \alpha$). Determine the character of the singularity (if any) at the origin. (Please assume that u is bounded; without some such hypothesis specified data would not determine u uniquely.)

(7) Let f and u be as in Problem 3. What condition on the Fourier series of f is equivalent to $\int_{x^2+y^2<1} |\nabla u|^2 \, dx \, dy < \infty$?

(8) Show that if u is harmonic on all R^n and $\int_{R^n} |u|^p \, dx < \infty$ for some $p > 1$, then u is identically zero. (Hint: start as we did in the proof of Liouville's theorem.)

(9) Suppose that u is harmonic on all R^n and $|u(x)| \leq C|x|^k$, where C is a constant and k is a positive integer. Show that u is a polynomial of degree at most k. (Hint: use induction on k, and ideas similar to those of our proof of Liouville's theorem.)