**PDE I – Problem Set 4.** Distributed Thurs 9/25/2014, due Tues 10/07/2014.

(1) Consider the heat equation $u_t = u_{xx}$ on $R$, with the "Heaviside function" as initial data:

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Show by integration against the fundamental solution that

$$u(x,t) = N(x/\sqrt{2t})$$

where $N$ is the cumulative normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, ds.$$

(b) Argue that this calculation is legitimate (i.e. $u$ solves the heat equation, and it has the desired initial data) although the Heaviside function is neither continuous nor compactly supported.

(2) Recall that for the heat equation in a bounded domain $\Omega$ with the Dirichlet boundary condition $u = 0$ at $\partial\Omega$, the solution decays exponentially to 0 as $t \to \infty$. Let's explore what happens in all space, focusing for simplicity on one space dimension:

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } t > 0,\ x \in R \\ u &= u_0(x) \quad \text{at } t = 0. \end{aligned}$$

(a) Show that if $u_0$ is bounded and continuous, and $\int_{-\infty}^{\infty} |u_0|\, dx < \infty$, then

$$\sup_x |u(x,t)| \le C t^{-1/2}.$$

What is the optimal value of $C$?

(b) Show that if $u_0 = \phi_x$ with $\int_{-\infty}^{\infty} |\phi|\, dx < \infty$ then the decay is faster:

$$\sup_x |u(x,t)| \le C t^{-1}.$$

What is the optimal value of $C$ in this case?

(3) Our discussion of the heat equation on the half-line $x > 0$ with a homogeneous Dirichlet ($u = 0$ at $x = 0$) or Neumann ($u_x = 0$ at $x = 0$) boundary condition used odd or even reflection. In particular, we used the following assertions:

- If $u_0 : R \to R$ is an odd function of $x$, then the solution of the whole-space heat equation with initial data $u_0$ is an odd function of $x$ for each $t$.
- If $u_0 : R \to R$ is an even function of $x$, then the solution of the whole-space heat equation with initial data $u_0$ is an even function of $x$ for each $t$.

(a) Give a proof of these assertions based on our solution formula (which gives $u(x,t)$ as the convolution of $u_0$ with the fundamental solution).

(b) Give a different proof of these assertions, based on the invariance of the PDE under reflection and the uniqueness of solutions to the initial value problem in all space. (You may assume for part (b) that $u_0$ is uniformly bounded, so that the uniqueness result of HW1 problem 7 is applicable.)

(4) Consider the heat equation in a the first quadrant of $R^2$, i.e.

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{for } x \in \Omega, t > 0 \\ u &= u_0 \quad \text{at } t = 0 \end{aligned}$$

with $\Omega = \{x_1 > 0,\ x_2 > 0\}$.

(a) Let $G(x, y, t)$ be the Green's function associated with the homogeneous Dirichlet boundary condition $u = 0$ at $\partial\Omega$. (By definition, this means that the solution of the PDE with this boundary condition has the form $u(x) = \int_\Omega G(x, y, t)u_0(y)\, dy$.) Give a formula for $G$.

(b) Let $H(x, y, t)$ be the Green's function associated with the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$. (By definition, this means that the solution of the PDE with this boundary condition has the form $u(x) = \int_\Omega H(x, y, t)u_0(y)\, dy$.) Give a formula for $H$.

(5) Let $\Omega$ be a bounded domain in $R^n$. The Neumann Green's function $N(x, y, t)$ is the analogue of the Dirichlet Green's function, but using the boundary condition $\partial u/\partial n = 0$ at $\partial\Omega$; its defining property is that the solution of $u_t - \Delta u = 0$ in $\Omega$ with $\partial u/\partial n = 0$ at $\partial\Omega$ and $u = u_0$ at $t = 0$ is $u(x, t) = \int_\Omega N(x, y, t)u_0(y)\, dy$. (Remark: $N(x, y, t)$ is symmetric in $x$ and $y$; the proof is parallel to what we did in class for the Dirichlet Green's function $G(x, y, t)$.) Show that the solution of

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{for } x \in \Omega, t > 0 \\ \partial u/\partial n &= g \quad \text{for } x \in \partial\Omega \\ u &= 0 \quad \text{at } t = 0 \end{aligned}$$

is given by

$$u(x, t) = \int_0^t \int_{\partial\Omega} N(x, y, t - s) g(y, s)\, dy\, ds.$$