PDE I – Problem Set 3. Distributed Thurs 9/18/2014, due Tues 9/30/2014.

(1) Let Ω be a bounded domain in R^n, and consider the semilinear heat equation $u_t - \Delta u = u^5$ in Ω, with $u = 0$ at $\partial\Omega$ and $u = u_0(x)$ at $t = 0$. Show that if $M_0 = \max_{x\in\Omega} |u_0(x)|$ and $M(t)$ solves $dM/dt = M^5$ with $M(0) = M_0$, then $|u(x,t)| \leq M(t)$ for all $x \in \Omega$ and all $t > 0$ such that both u and M exist. [Hint: use the maximum principle. Remark: One can show that if u ceases to exist at a finite time $T_{\max}$, then $\max_{x\in\Omega} u(x,t) \to \infty$ as $t \to T_{\max}$. Therefore this problem gives a lower bound for $T_{\max}$.]

(2) Suppose u solves

$$u_t - u_{xx} = 16u$$

on the interval $(0, \pi)$, with the homogeneous Neumann condition $u_x = 0$ at $x = 0, \pi$. Characterize the initial data $u_0 = u(x, 0)$ for which $u(x,t)$ stays bounded as $t \to \infty$.

(3) The Lecture 2 notes discuss a continuous-time, discrete-space approximation of $u_t = u_{xx}$. When u has a homogeneous Dirichlet boundary condition, the ODE for the nodal values is

$$\dot{u}_j = \frac{u_{j-1} + u_{j+1} - 2u_j}{(\Delta x)^2} \quad j = 1, \dots, N-1$$

with the convention that the domain is $(0, N\Delta x)$ and $u_0(t) = u_N(t) = 0$. Let's discuss its convergence as $\Delta x \to 0$.

(a) Suppose the exact solution has u^{pde}_{xxxx} bounded (uniformly with respect to space and time). Consider the error $z_j(t) = u_j(t) - u^{\text{pde}}(j\Delta x, t)$. Show that if we define $\phi_j(t)$ by

$$\dot{z}_j - \frac{z_{j-1} + z_{j+1} - 2z_j}{(\Delta x)^2} = \phi_j(t), \tag{1}$$

then we have an estimate of the form $|\phi_j| \leq C(\Delta x)^2$, with the constant C depending only on an upper bound for $|u^{\text{pde}}_{xxxx}|$.

(b) Show that if the RHS of (1) were zero we would have a discrete version of the maximum principle. In other words: show that if $w_j(t)$ $(j = 1, \dots, N-1)$ solves the ODE system $\dot{w}_j - \frac{w_{j-1}+w_{j+1}-2w_j}{(\Delta x)^2} = 0$ with the convention $w_0(t) = w_N(t) = 0$, then $\max_{j,t} w_j(t)$ and $\min_{j,t} w_j(t)$ are achieved either at the initial time $(t = 0)$ or the spatial boundary $(j = 0$ or $j = N)$.

(c) Apply part (b) to $z_j \pm C(\Delta x)^2 t$ to deduce the error estimate $|z_j(t)| \leq C(\Delta x)^2 t$.

(4) The Lecture 2 notes also discuss a discrete-time, discrete-space approximation of $u_t = u_{xx}$. When u has a homogeneous Dirichlet boundary condition and the domain is $(0, \pi)$, the scheme says

$$u_j(t_{n+1}) = \alpha u_{j+1}(t_n) + \alpha u_{j-1}(t_n) + (1 - 2\alpha)u_j(t_n) \tag{2}$$

with the conventions that the spatial step is $\Delta x = \pi/N$, the times are $t_n = n\Delta t$,

$$\alpha = \frac{\Delta t}{(\Delta x)^2},$$

and $u_0(t_n) = u_N(t_n) = 0$ for all n. I told you that this scheme is stable for $\alpha \leq 1/2$ and unstable for $\alpha > 1/2$. Let's understand why.

(a) Assume $0 < \alpha \le 1/2$. Show that for any M, if initially $\max_j |u_j(0)| \le M$, then the estimate persists: $\max_j |u_j(t_n)| \le M$ for each $n = 1, 2, \ldots$. (Thus, the scheme is stable in the sense that a small change in its initial data produces a small change in the solution.)

(b) Suppose $\alpha > 1/2$. Consider, for any integer k, the initial data $u_j(0) = \sin(jk\Delta x)$. (Note that it vanishes at the endpoints $j = 0, N$.) Show that the associated solution is

$$u_j(t_n) = \xi^n u_j(0)$$

where $\xi = \xi(k) = 1 - 2\alpha[1 - \cos(k\Delta x)]$.

(c) The solution identified in part (b) grows exponentially in magnitude if $|\xi| > 1$. Show that if $\alpha > 1/2$, then such growth happens when $\cos(k\Delta x)$ is close enough to -1. (Thus, the scheme is unstable in the sense that a small change in its initial data can produce a huge change in the solution after multiple time steps, even at times such that $t_n = n\Delta t$ is still quite small.)

(5) The Lecture 3 notes show that if Ω is a bounded domain, the linear heat equation $u_t - \Delta u = 0$ with boundary condition $u = 0$ at $\partial\Omega$ can be interpreted (using the L^2 inner product) as "steepest descent" for $F[u] = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$ within the class of functions satisfying $u = 0$ at $\partial\Omega$. Let's give similar interpretations to some nonlinear PDE's. In both parts (a) and (b), the domain Ω is bounded and the boundary condition is $u = 0$ at $\partial\Omega$.

(a) Show that $u_t = \Delta u + u^5$ is "steepest descent" (with respect to the L^2 inner product, and within the class of functions that vanish at $\partial\Omega$) for $G[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{6}u^6\,dx$.

(b) Show that $u_t = \mathrm{div}\,(|\nabla u|^2 \nabla u)$ is "steepest descent" (with respect to the L^2 inner product, and within the class of functions that vanish at $\partial\Omega$) for $H[u] = \int_\Omega \frac{1}{4}|\nabla u|^4$.

(6) Show that if a pde has a time-periodic solution then it cannot have a steepest-descent interpretation.

(7) In Lecture 3 we showed that the "gradient" of $F[u] = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$ in the class of functions satisfying $u = 0$ at $\partial\Omega$ (and using the L^2 inner product) is $-\Delta u$. Now let's consider the same functional, but without imposing any boundary condition on u. Show that in this case ∇F exists at u only if $\partial u/\partial n = 0$ at $\partial\Omega$. (Thus: the linear heat equation with the Neumann boundary condition $\partial u/\partial n = 0$ at $\partial\Omega$ represents steepest descent for F in the class of functions with no specified boundary condition.)

(8) The Lecture 3 notes discuss "implicit-in-time" discretization of the linear heat equation. [We didn't get to this in class on 9/16; we'll spend a few minutes on it at the start of class 9/23.] Focusing on the discrete-time, continuous-space setting with $u = 0$ at the boundary, this scheme chooses $u(x, t_{n+1})$ by solving

$$\frac{u(x,t_{n+1}) - u(x,t_n)}{\Delta t} = \Delta u(x, t_{n+1})$$

with $u(x, t_{n+1}) = 0$ at $\partial\Omega$. (Please accept that such $u(x, t_{n+1})$ exists.) Show that $\int_\Omega |\nabla u(x, t_{n+1})|^2\,dx \le \int_\Omega |\nabla u(x, t_n)|^2\,dx$. [Remark: this no surprise, since in the continuous-time setting the Dirichlet integral is monotonically decreasing. But it provides a first indication of the implicit scheme's robust stability, since we made no assumption about the size of Δt.]