PDE I – Problem Set 2. Distributed Wed 9/10/2014, due Tues 9/23/2014.

(1) The last part of the Lecture 1 notes states and proves a "maximum principle" for the heat equation in a bounded domain with a Dirichlet boundary condition. This problem asks you to do something similar for a Neumann boundary condition. Suppose $u_t - \Delta u = 0$ on $\Omega \times (0,T)$ with $\partial u/\partial n = 0$ at $\partial\Omega$. Show that u achieves its maximum and minimum values at the initial time $t = 0$. [Hint: consider $u_{\epsilon,\delta}(x) = u - \delta\phi(x) - \epsilon t$, with a suitable choices of $\phi(x), \delta$, and ϵ. You may assume that $\partial\Omega$ is as smooth as you like.]

(2) Let's look at how the maximum principle changes when the PDE has a zeroth order term. Throughout this problem, we work in a bounded domain $\Omega \subset R^n$, with Dirichlet boundary condition $u = 0$ at $\partial\Omega$ and initial condition $u(x,0) = u_0(x)$.

(a) Suppose the PDE is

$$u_t - \Delta u + c(x,t)u = 0$$

with $c(x,t) \geq 0$. Show that

$$\max u \leq \max u_0^+ \quad \text{and} \quad \min u \geq \min u_0^-$$

where u_0^+ and u_0^- are respectively the positive and negative parts of u_0.

(b) Consider the same PDE, but assume now that $c(x,t) \geq \gamma$ where γ is a positive constant. Show that $|u(x,t)| \leq Ce^{-\gamma t}$. [Hint: apply part (a) to $ue^{\gamma t}$.]

(c) Consider the same PDE, but let $c(x,t)$ be any smooth function (bounded, but possibly negative). Show that if $u_0 \geq 0$ then $u(x,t) \geq 0$ for all $x \in \Omega$ and $t > 0$. [Hint: consider $v(x,t) = e^{\lambda t}u(x,t)$ for a suitable choice of λ.]

(3) Consider two solutions u_1 and u_2 of the semilinear parabolic equation

$$u_t - \Delta u = f(u)$$

in a bounded domain Ω, with the same Dirichlet boundary data but different initial conditions. Show that if initially $u_1(x,0) \leq u_2(x,0)$ for all $x \in \Omega$, then this property holds for all time: $u_1(x,t) \leq u_2(x,t)$ for all $x \in \Omega$ and all $t > 0$. [Hint: show that $w = u_2 - u_1$ solves an equation of the form $w_t - \Delta w = c(x,t)w$.]

(4) Let u solve the semilinear equation

$$u_t - \Delta u = f(u)$$

in a bounded domain Ω, with a Dirichlet boundary condition $u = 0$ at $\partial\Omega$. Suppose $u_t \geq 0$ initially (in other words, suppose $\Delta u_0 + f(u_0) \geq 0$, where u_0 is the initial condition). Show that $u_t \geq 0$ for all $x \in \Omega$ and all $t > 0$. [Hint: start by differentiating the equation in time, to get a PDE satisfied by u_t.]

(5) We briefly discussed in class how our proof of the maximum principle extends to equations of the form $u_t - \sum a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x,t)\frac{\partial u}{\partial x_i} = 0$, provided that for every (x,t) the matrix $a_{ij}(x,t)$ is symmetric and nonnegative. A key step is the observation that if u has a local minimum at an interior point $(x_0,t_0) \in \Omega \times (0,T)$ then $\sum a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \geq 0$ at (x_0,t_0). Explain why this is true.

(6) Show that if $\Omega \subset R^n$ is bounded and u_0 is given, there can be at most one solution of the nonlinear boundary value problem $u_t - \Delta u + |\nabla u|^2 = 1$ in $\Omega \times (0,T)$ with $u = u_0$ at $t = 0$ and $u = u_0$ at $\partial\Omega$. (Hint: if there are two solutions, let w be the difference. Show that it solves an equation of the form $w_t - \Delta w + \sum b_i(x,t)\frac{\partial w}{\partial x_i} = 0$.)

(7) We briefly discussed in class a version of the maximum principle for a parabolic PDE in all R^n. This problem asks you to work out the details.

   (a) Suppose $u_t - \Delta u \le 0$ for $x \in R^n$ and $t \in (0,T)$, and assume furthermore that u is globally bounded: $|u(x,t)| \le C$ for all x, t, where C is a constant. Show that u achieves its maximum at the initial time $t = 0$. (Hint: consider $u_{\varepsilon,\delta} = u - \varepsilon|x|^2 - \delta t$.)

   (b) Would the same argument work if instead of uniform boundedness we assume the weaker condition $|u| \le C(1 + |x|)$? What about if we replace the hypothesis $u_t - \Delta u \le 0$ by $u_t - \sum a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x,t)\frac{\partial u}{\partial x_i} \le 0$?

The remaining problems are concerned with the solution of $u_t - u_{xx} = 0$ on the interval $(0,\pi)$ with the homogeneous Dirichlet boundary condition $u(0,t) = u(\pi,t) = 0$ and initial condition $u_0(x)$. The advantage of working in 1D is that we know the eigenvalues and eigenfunctions of the Laplacian explicitly. When specialized to this setting, our solution formula becomes

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x) \tag{1}$$

with $\phi_n(x) = \sqrt{2/\pi}\sin(nx)$ and $a_n = \int_0^\pi u_0(x)\phi_n(x)\,dx$.

(8) Suppose the coefficients a_n in (1) are uniformly bounded, in other words $|a_n| \le C$ for all n. (Note that this does not imply convergence of $\sum a_n\phi_n$.) Show that the function $u(x,t)$ defined by (1) is C^∞ in x for each $t > 0$.

(9) Assume now that $u_0(x)$ has two derivatives, with $|u_0''(x)| \le M$ for some constant M. Assume further that u_0 satisfies the boundary condition, i.e. $u_0(0) = u_0(\pi) = 0$.

   (a) Prove an inequality of the form $|a_n| \le C/n^2$.

   (b) Show that as t decreases to 0, the function $u(x,t)$ defined by (1) converges uniformly to $u_0(x)$.

(10) Suppose now that $u_0(x)$ has four bounded derivatives ($|u_0''''(x)| \le M$), and $u_0''(0) = u_0''(\pi) = 0$.

   (a) Show that as t decreases to 0, $u_{xx}(x,t)$ converges uniformly to $u_0''(x)$.

   (b) If u_0 is smooth but $u_0''(0) \ne 0$ or $u_0''(\pi) \ne 0$, is it possible that the conclusion of part (a) still holds?

(11) The solution formula (1) makes sense even when u_0 doesn't vanish at the endpoints – for example when $u_0(x) \equiv 1$.

   (a) Does $u(x,t)$ satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$?

   (b) Discuss the sense in which $u(x,t)$ approaches u_0 as $t \downarrow 0$ in this case.