PDE I – Problem Set 12. Distributed 12/3/2014, due in class 12/9/2014. *The maximum permitted extension will be Thurs 12/11 by 6pm. A solution sheet will be distributed soon after that.*

Note: Our final exam is Tues 12/16 in the normal class location and time. You may bring *two* sheets of notes (8.5×11, both sides, any font). The exam is cumulative: its scope includes everything done up to and including the 12/2 lecture. (On 12/2 we stopped around page 6 of the Lecture 12 notes; that is, we discussed the method of characteristics but not the other material in those notes. So: the method of characteristics is included in the scope of the exam, but the other material in the Lecture 12 notes is not.)

(1) A "viscous shock profile" for Burgers' equation is a function $U(\xi)$ with $U \to u_L$ as $\xi \to -\infty$ and $U \to u_R$ as $\xi \to +\infty$, such that $u(x,t) = U[(x-\sigma t)/\epsilon]$ solves $u_t + uu_x = \epsilon u_{xx}$. We know that if U exists then σ must be equal to $(u_L + u_R)/2$ and we must have $u_L > u_R$.

(a) Show that U does indeed exist, and it's given by

$$\begin{aligned}\xi &= \int \frac{dU}{\frac{1}{2}(U-u_L)(U-u_R)} \quad + \text{constant} \\ &= \frac{2}{u_L - u_R} \log\left(\frac{u_L - U}{U - u_R}\right) \quad + \text{constant}\end{aligned}$$

(b) Show that as $\epsilon \to 0$ (and choosing the constant properly) $\lim_{\epsilon\to 0} U[(x-\sigma t)/\epsilon]$ converges to the admissible weak solution of Burgers' equation with initial condition u_L for $x < 0$ and u_R for $x > 0$.

(c) What happens to the above construction if $u_R > u_L$ (so that no shock is expected)?

(2) Use the method of characteristics to solve these initial value problems:

(a) $u_t + u_x = u^2, \quad u(x,0) = h(x)$

(b) $u_t + xu_x + yu_y = u, \quad u(x,y,0) = h(x,y)$

(c) $xu_t - tu_x = u, \quad u(x,0) = h(x)$.

(3) Consider the "equation of geometrical optics" in 2D:

$$u_x^2 + u_y^2 = 1.$$

(a) Find the associated system of ODE's describing the characteristics.

(b) Show that ∇u is constant along each characteristic.

(c) Suppose $u = 0$ on a smooth, simple closed curve S in the x, y plane. Show that $u(x) = \pm$distance to S, if u is differentiable along the shortest path from x to S.

(d) Show that u cannot be smooth throughout the bounded region determined by S.

(4) Consider the first-order PDE $u_t + H(x, \nabla u) = 0$ (this "Hamilton-Jacobi equation" arises naturally in the context of optimal control, as we'll discuss on 12/9). Show that solving this PDE by the method of characteristics leads to the ODE system

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad \dot{u} = -H + \sum_i p_i \frac{\partial H}{\partial p_i}$$

with the notation $H = H(x,p)$. (If you know some physics, you'll recognize that the equations for $x(t)$ and $p(t)$ are those of Hamiltonian mechanics.)