PDE I – Problem Set 11. Distributed 11/26/2014, due in class 12/9/2014. *The maximum permitted extension will be Thurs 12/11 by 6pm. A solution sheet will be distributed soon after that.*

End of semester plan: Our last two lectures are 12/2 and 12/9. On 12/2 we will finish scalar conservation laws then discuss the method of characteristics more generally. A final (relatively short) problem set will be distributed on 12/2; for it, too, the maximum permitted extension will be Thurs 12/11 by 6pm, and a solution sheet will be distributed shortly thereafter. The last (12/9) lecture will be on Hamilton-Jacobi equations; that material will not be on the final exam. Our exam is Tues 12/16 in the normal class location and time.

Note: The 11/25 lecture stopped at about page 14 of my Lecture 11 notes (after defining the notion of an admissible shock, but before discussing how this is related to viscous shock profiles). Some of the following problems require you to find explicit solutions of scalar conservation laws. Your solutions should of course use only admissible shocks. These problems require only the material we covered on 11/25.

(1) We know that

$$u(x,t) = \begin{cases} 2 & \text{for } x < 3t/2 \\ 1 & \text{for } x > 3t/2 \end{cases}$$

is an admissible weak solution of Burgers' equation (the shock speed is $(u_L + u_R)/2 = 3/2$). Show that u is not a weak solution of the conservation law $\frac{1}{2}(u^2)_t + \frac{1}{3}(u^3)_x = 0$. (Note that for smooth nonzero u, this conservation law reduces – like Burgers' equation – to $u_t + uu_x = 0$.)

(2) Consider Burgers' equation with initial data

$$u(x,0) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 < x < 1/2 \\ 1-x & \text{for } 1/2 < x < 1 \\ 0 & \text{for } x > 1. \end{cases}$$

Find the time and place of shock formation, and the velocity of the shock for all later times.

(3) Consider Burgers' equation with initial data

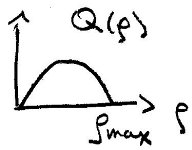
$$u(x,0) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 < x < L \\ 0 & \text{for } x > L. \end{cases}$$

Show that when t is large enough the solution has the form

$$u(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ x/t & \text{for } 0 < x < s(t) \\ 0 & \text{for } x > s(t) \end{cases}$$

and determine the shock locus $s(t)$. (Hint: one method for finding $s(t)$ uses the Rankine-Hugoniot condition combined with the fact that $\frac{d}{dt}\int u\,dx = 0$.)

(4) As explained in the Lecture 11 notes, a continuum model of traffic flow on a 1D road is $\rho_t + (Q(\rho))_x = 0$ where Q (the traffic flux, with units cars per unit time) should vanish at $\rho = 0$ and at $\rho = \rho_{\max}$ and be unimodal in between, as in the figure.



(a) Consider the solution with initial data

$$\rho = \begin{cases} \rho_{\max} & x < 0 \\ 0 & x > 0 \end{cases}$$

(This captures what happens after a red light turns green, if the backup behind the light is infinite.) Discuss both the space-time picture (is the solution a fan or a shock?) and the form of the function $x \mapsto \rho(x,t)$ at a given time.

(b) Let ρ_1 and ρ_2 be constants, chosen so that ρ_1 is below the value where Q is maximized, ρ_2 is above the value where Q is maximized, and $Q(\rho_1) = Q(\rho_2)$. (Note that $Q'(\rho_1) > 0$ and $Q'(\rho_2) < 0$.) Discuss the solution with initial data

$$\rho = \begin{cases} \rho_1 & x < -1 \\ \rho_2 & -1 < x < 0 \\ \rho_1 & x > 0 \end{cases}$$

(The special case $\rho_1 = 0, \rho_2 = \rho_{\max}$ captures a red light turning green, if the backup behind the light is finite.) As for (a), you should discuss both the spacetime picture (hint: there is both a fan and a shock) and the form of the function $x \mapsto \rho(x,t)$. (Note: there is a qualitative change in the form of this function at a certain time).

(5) [Problem 8 of Guenther & Lee Section 12.3] Consider the variable-coefficient wave equation $u_{tt} = (a(x,t)u_x)_x$, where $a(x,t)$ is a given function (assumed strictly positive and smooth). The associated conservation law is

$$\frac{d}{dt}\int_\alpha^\beta u_t(x,t)\,dx = a(\beta,t)u_x(\beta,t) - a(\alpha,t)u_x(\alpha,t)$$

for all $\alpha < \beta$.

(a) Suppose u is continuous but only piecewise smooth, with first derivatives that may be discontinuous along a curve $x = \sigma(t)$. Show that for the conservation law to hold, u must satisfy the PDE on both sides of the "shock" $x = \sigma(t)$, and σ must satisfy

$$[u_t]\dot{\sigma} + a(\sigma(t),t)[u_x] = 0,$$

where the square bracket [f] denotes the the jump of f across the "shock" $x = \sigma(t)$ (i.e. $[f] = f_R - f_L$ where f_R and f_L are the values of f for x just above and below $\sigma(t)$).

(b) Differentiate $u(\sigma(t), t)$ to show that u_L and u_R satisfy

$$\begin{aligned}
\tfrac{d}{dt}u(\sigma(t), t) &= (u_L)_x(\sigma(t), t)\dot{\sigma} + (u_L)_t(\sigma(t), t) \\
\tfrac{d}{dt}u(\sigma(t), t) &= (u_R)_x(\sigma(t), t)\dot{\sigma} + (u_R)_t(\sigma(t), t).
\end{aligned}$$

Conclude that $[u_t] + [u_x]\dot{\sigma} = 0$.

(c) Suppose that at least one of the jumps $[u_t]$ or $[u_x]$ is nonzero. Use the preceding results to deduce that $\dot{\sigma}^2 = a(\sigma(t), t)$. (Thus: this wave equation tolerates singular "weak solutions", but they they must move with velocity $\pm\sqrt{a}$).